Previous Page

usually less expensive than their macroscopic counterparts. The development of micromechanical systems, however, requires appropriate fabrication technologies which permit the definition of small geometry, precise dimensional control, design flexibility, interfacing with control electronics, high repeatability and reliability, high yield, and low cost.

See Plate 42.

See also: **MEMS, applications**; **MEMS, dynamic response**.

Further Reading

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MODAL ANALYSIS, EXPERIMENTAL

Basic principles

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Overview

This article introduces and explains the basic concepts of the widely-used technology which has become known as 'experimental modal analysis' or 'modal testing'. Simply put, experimental modal analysis comprises a set of experimentally-based procedures which lead to the construction of a mathematical model that can be used to describe the dynamic behavior of the test object. This model can be used on a variety of useful applications including:

1. visualization of the modes of vibration of the test structure for the purpose of gaining physical insight and an understanding of the often complex dynamic properties of real structures;
2. comparison of the actual (measured) vibration behavior of a real structure with corresponding parameters predicted from a theoretical model;
3. correcting or refining that theoretical model;
4. predicting the effects of making modifications, or predicting what modifications to introduce to change the structure's behavior;
5. predicting the behavior of structures formed by coupling two or more components together;
6. detecting damage or other changes in the integrity of a structure during its service life;
7. identification of 'unpredictable' parameters such as damping, dynamic friction effects, excitation forces from unknown sources, etc.

In effect, modal testing has been practiced for a long time but has only been formalized in the past 30 years, following the advent of computers and instrumentation capable of realizing the concepts with sufficient accuracy to warrant implementation of the ideas.

The basic sequence of steps in a modal test are:

1. measurement of the test structure's vibration response to a controlled and known excitation;
2. analysis of the resulting response functions to identify the underlying modal properties (natural frequencies and mode shapes) of the test structure;
3. construction of a mathematical model from these modal properties, suitable for the intended application.

The specific skills that are required to conduct a successful modal test include:

- a sound appreciation of the underlying theory
- measurement techniques to excite and measure the vibration response of the test structure
- signal processing techniques to define the required response function information
- data analysis methods which are capable of extracting parameters for an assumed mathematical model that can describe the observed test behavior.

Brief History of Experimental Analysis/ Modal Testing

Modal testing has been practiced at various levels for over 50 years. The earliest, and still the most common, applications of the experimental methods that are now called 'modal testing' or 'experimental modal analysis' were to validating whatever theoretical model had been constructed to predict the structure's behavior under dynamic loading conditions – usually under steady-state excitation. Nowadays, the mathematical models in question are almost always finite element models. Probably the first documented modal testing methods date back to the 1940s when engineers charged with conducting vibration tests on aircraft structures† sought to supplement the primary test data of how and when the structure would fail under dynamic loads with intermediate observations (while the test was running) of the input forces and the resulting response levels, thereby measuring the same type of frequency response functions that we use today. These early attempts to conduct what we now recognize as modal tests were frustrated by inadequacies of the transducers and signal conditioning equipment of the day and it was a full decade later before early electronic devices, and then computers, provided the means to make and to analyze adequate measurements of the essential data. In fact, it was the widespread introduction of the digital computer and development of data analysis techniques such as the fast Fourier transform (FFT) in the 1960s that heralded the real start of modern experimental modal analysis.

The 1970s saw the development and widespread use of FFT-based signal analyzers and the associated progress of the technology necessary to process the resulting measured data by extracting the coefficients in an assumed governing mathematical model. By the 1980s, experimental modal analysis was maturing to the stage where it could be used in a routine industrial context and it was during this decade that many of the powerful applications were developed: model updating; structural dynamic modification (SDM) and optimization methods as well as those for analyzing structures which comprise an assembly of different and often disparate components; and then later, damage detection, and identification of a range of quantities that are difficult, if not impossible, to describe theoretically.

More recent developments in the subject, in the 1990s, have seen progressive refinements in the primary techniques and applications, accompanied by a growing appreciation of the limitations associated by almost any experimental data in part by their inevitable imprecision but also, and more significantly, by their incompleteness, a feature brought about by the inescapable restrictions on the quantity and choice of what can be included in a given test. Bearing these effects in mind, and the trend towards the conduct of fewer and fewer tests, not least because of the resources and time they consume, there is now considerable interest and activity in refining modal test procedures and methods so as to optimize the whole process: to make careful selection of which parameters must be measured (and which are unimportant) and the accuracy that is required. By such an approach, the cost-effectiveness of modal tests can be considerably increased and their usefulness assured. These recent developments are often referred to as 'test planning' and represent a 'full circle' situation as regards the theoretical models that most modal tests are performed to validate: the subject theoretical models are often used in a numerical simulation of the proposed modal test in order to establish the optimum parameters for that test.

Basic Procedures Involved in a Modal Test

The sequence of steps that must be executed to conduct a modal test (the engineer having first thoroughly familiarized him/herself with the associated theory so that he/she can detect and explain any deviations of actual behavior from the expected and assumed characteristics of a linear, multi-degree-of-freedom system (MDOF)) are as follows:

1. set up the test structure in a mounting configuration which has been carefully selected and which can be controlled;
2. provide a means of exciting the structure into vibration in a controlable and measurable way. This may be by means of a nonattached device, such as an impact hammer or similar, or by an exciter which is connected firmly to the test structure. These exciters can generate the required

† At that time, 'vibration tests' were those conducted to simulate service loading and to determine the threshold levels that the structure could withstand before failing. They were, in effect, environmental of endurance tests.

excitation forces by several means: mechanical, electromagnetic, electrohydraulic, etc.;
3. provide a means of (transducers for) measuring the resulting response of the test structure, and to do so with the minimum of interference to the test object (often difficult to achieve);
4. provide signal processing and analysis facilities so that the required information can be extracted from the individual measured time histories yielded by the transducers. Here, it is usually necessary to convert raw measured data in the time domain into equivalent spectra in the frequency domain, as frequency domain parameters are more commonly used to describe most vibration phenomena;
5. subject the measured response function data to a subsequent analysis stage, often employing curve-fitting techniques, in order to construct a mathematical model of standard form (linear, MDOF system) whose dynamic properties most closely resemble those observed on the test structure.
6. Check that the resulting model is adequate for the intended application (before releasing the test structure).

The Essential Theory

It is important in modal testing, more so than in many experimental disciplines, that the underlying theory of both the structural dynamic behavior and the various stages of signal processing and data analysis (typically, curve-fitting) is well appreciated by those who conduct such tests. There are many situations in which poor, and even very poor, results can be obtained using perfectly adequate equipment and instrumentation because these are not being used correctly: a situation which can easily arise if the understanding of the whys and wherefores of the different techniques involved is inadequate.

Much of the necessary theory can be found in more detail in other entries in this work (see **Modal analysis, experimental**, Parameter extraction methods; **Modal analysis, experimental**, Construction of models from tests). Accordingly, here we shall simply quote some of the more important expressions and formulas and indicate some of those areas that should be mastered as necessary preparation for those aspiring to conduct successful modal tests.

Having described experimental modal analysis as the process whereby a mathematical model is constructed from test data which is capable of describing the observed behavior of the test structure, it is appropriate here to define more in detail what is meant by a mathematical model. Conventionally, there are three types of mathematical model, each of which is capable of describing the required structural dynamic behavior, but each formulating the model in a different way, based on different fundamental features.

Spatial Models

The first type of model is the spatial model which describes the distribution in space of the essential physical features of the structure – its mass or inertia, its stiffness and its damping properties. In simple terms, the spatial model is the set of mass, stiffness and damping elements which are used in describing equations of motions for the system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

or

$$\mathbf{M}\mathbf{x} + \mathbf{C}\mathbf{x} + \mathbf{K}\mathbf{x} = \mathbf{f}$$

for the more representative MDOF system that is applicable to most real cases.

Modal Models

The second type of model is the modal model, which comprises the set of eigenvalues and eigenvectors – natural frequencies and mode shapes – that describe the individual 'ways' in which the structure will choose to vibrate if left to do so without any externally applied excitation, other than an initial disturbance from which it is released and allowed to vibrate. This model is defined by two matrices, $\boldsymbol{\omega}_r^2$ and $\boldsymbol{\Phi}$, containing the eigenvalues and eigenvectors, respectively.

There is a direct and algebraically simple relationship between the spatial model and the modal model which can be written for the basic undamped case as follows:

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}$$

and:

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi} = \bar{\boldsymbol{\omega}}_r^2$$

Response Models

The third type of model is referred to as a response model, and this consists of a set of response functions (usually but not exclusively, frequency response functions or FRFs) that relate the input/output relationships for all the degrees of freedom of the structure. The typical individual FRF from which this model is built is written as $H_{jk}(\omega)$, and is defined as the harmonic response in DOF j to a unit harmonic excitation applied in DOF k at frequency, ω. There is an important condition associated with this definition, which is that the specified excitation must be the only excitation that is applied when the response is measured. Thus:

$$H_{jk}(\omega) = \left(\frac{X_j}{F_k}\right); \quad F_m = 0; \quad m = 1, N; \neq k$$

In fact, the complete FRF matrix, $\mathbf{H}(\omega)$, which contains the full set of individual FRFs for all j, k combinations, can be related to the preceding spatial and modal models as follows (again, shown here for the basic undamped system):

$$\mathbf{H}(\omega) = \left(\mathbf{K} - \omega^2\mathbf{M}\right)^{-1} = \mathbf{\Phi}\left[\left(\bar{\omega}_r^2 - \omega^2\right)\right]^{-1}\mathbf{\Phi}^{\mathrm{T}}$$

From this last expression, it is possible to derive a convenient expression for the individual FRF as:

$$H_{jk}(\omega) = \sum_{r=1}^{N} \frac{(\phi_{jr})(\phi_{kr})}{\bar{\omega}_r^2 - \omega^2}$$

and this is the expression which forms the basis for much of modal analysis processing methods.

See **Modal analysis, experimental**, Measurement techniques for further details of this underlying theory for experimental modal analysis.

Summary of Essential Experimental Requirements

The experimental aspects of modal testing comprise two main tasks: (i) exciting the test structure into vibration in a controled way and (ii) measuring the excitation force(s) and the resulting response(s). These tasks embrace a series of mechanical issues of excitation devices, support of the test structure, attachment of exciters and transducers plus another set of requirements concerning the signal conditioning and processing which must be applied to the transducer outputs. A sketch of a typical set-up for conducting a modal test is shown in **Figure 1**, with the major elements of the system labeled appropriately.

Support of Test Structure

The first task is to decide how to support the test structure. There are essentially two choices, free–free or grounded and the selection depends on a number of factors which are determined by the eventual application of the test results. Careful consideration must be given to these and the appropriate choice must be made.

Excitation of Test Structure

Next, it is necessary to decide whether the excitation forces should be applied by an attached exciter, or by a nonattached impact device, such as a hammer. A major advantage of the former is that much more control can be applied to the excitation levels and frequency content than is possible with an impactor. On the other hand, the very attachment of the exciter to the structure introduces the risk of interfering with the structure and changing its local stiffness or mass. Care must therefore be taken to attach the exciter in a well-controlled way. The use of a hammer exciter, or equivalent, has the advantage of being

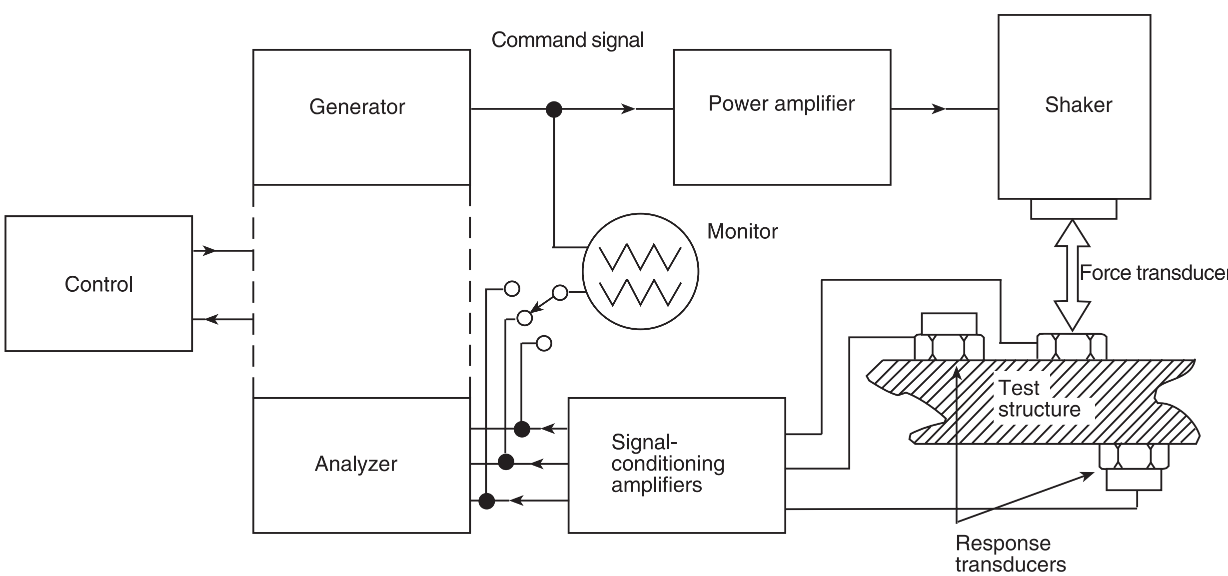


Figure 1 Typical experimental set-up for modal test.

very convenient and quick, as no attachment device is required. However, relatively little control of the excitation signal and level is possible with such an exciter and it is found that there is a relatively narrow range of structures which are well suited to impact excitation – not too lightly-damped; not too heavily damped; linear – while those outside that set can pose problems.

The structure can be excited with almost any type of signal – sinusoidal, periodic, random, transient – and essentially the same response function will be derived from each. There has been a lot of development to determine the most effective and efficient excitation signals and the choice of the one to use in each case needs careful consideration. **Figure 2** shows some of the more popular signal types.

Transducers

A wide range of transducers are available for capturing the excitation force and resulting response levels. Many of these are piezoelectric devices, simple to use and relatively free of idiosyncratic features. Such transducers are widely available to measure force and acceleration: other transducers are also available for measuring velocity responses directly, and worth a special mention are the new generation of laser Doppler vibrometers, which offer

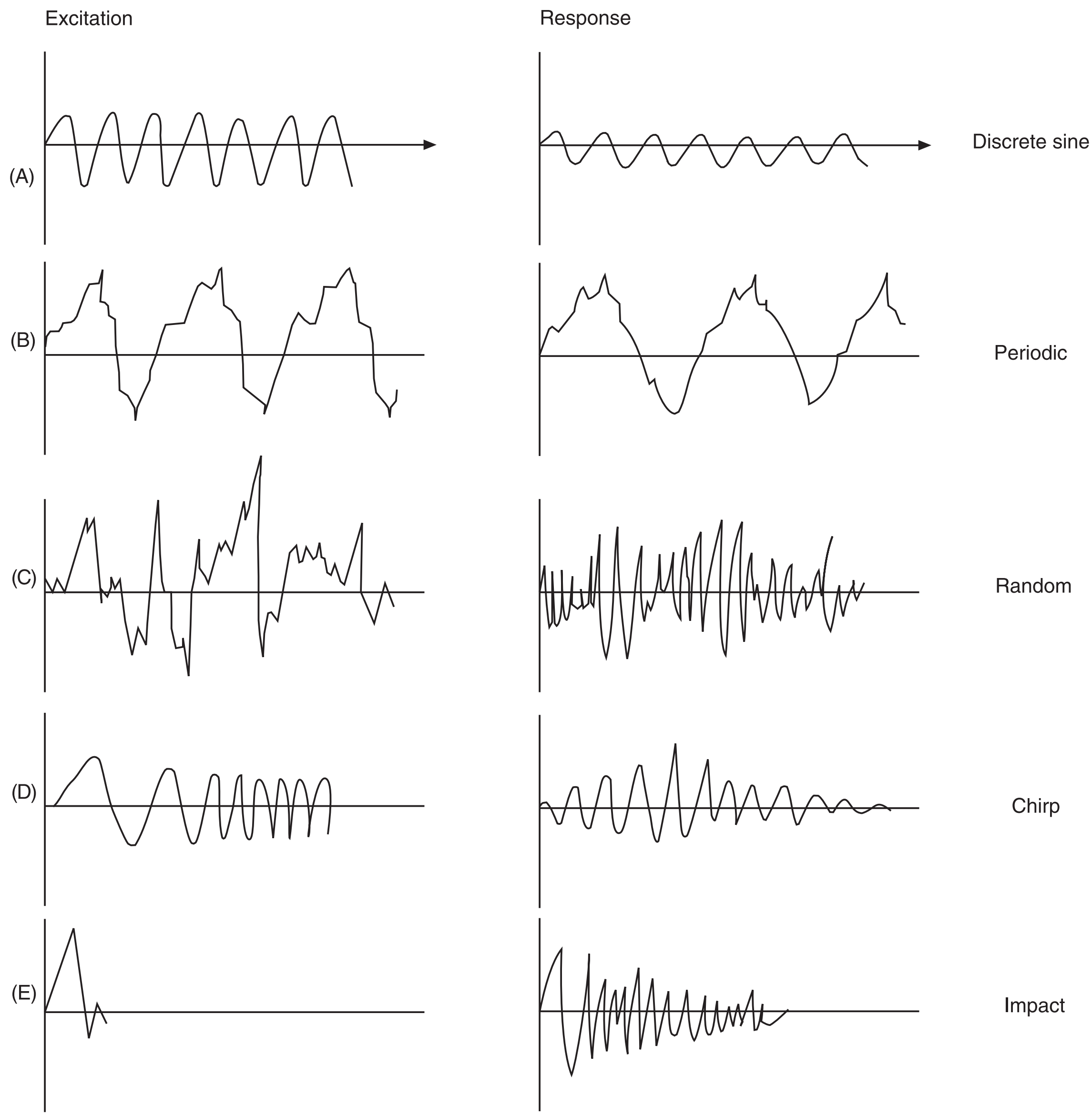


Figure 2 Popular types of excitation signal used for modal tests. (A) discrete sine; (B) periodic; (C) random; (D) transient–chirp (E) transient–impact.

advantages of being noncontacting devices, thereby minimizing interference with the behavior of the test structure.

Signal Processing

Signal processing is a major activity. It is generally required to be able to extract the individual frequency components which are present in a signal – sometimes because the original signal generating the vibration contains many components (as is the case for periodic, random or transient types of excitation), and other times because it is required to eliminate spurious components of response, introduced by noise or nonlinearities in the measuring system. These various requirements can be met with the current generation of spectrum and other frequency response analysers, often based on digital filtering and frequently employing the FFT or similar algorithms developed in the 1960s, and making signal processing very much faster than it had been hitherto. The advent of the FFT was a major development for modal testing.

The primary output from the measurement stage of a modal test consists of a series of response functions, usually – but not exclusively – FRFs, and these are yielded directly from the output of the signal processing devices (analyzers) used to treat the measured time-histories emitted from the transducers. Here, again, a thorough appreciation of the underlying theory becomes essential for the successful use of the various experimental and signal processing devices. **Figure 3** illustrates some typical signals recorded during a modal test, and the resulting FRF that is produced as the 'output' from the measurement phase of the test.

See **Modal analysis, experimental**, Measurement techniques for details of these experimental aspects of conducting a modal test.

Summary of the Analysis of Measured Response Functions

The next stage in the modal test procedure is the analysis which is applied to the measured response functions in order to reveal the properties of the mathematical model which closely describes the behavior of the measured structure.

The first part of this stage comprises a process of 'modal parameter extraction' or 'modal analysis' in which the measured response functions are scrutinized in some detail in order to determine the underlying formulas which can be used to characterize their form. This is usually achieved by using curve-fitting procedures, where the basis of each curve is a theoretically-generated formula developed for an MDOF system with, at the outset, unknown parameters. By applying curve-fitting procedures, least-squares or other criteria can be applied in order to determine the best-fit coefficients in these expressions such that a theoretically-regenerated curve can be constructed to pass through the measured data points with the minimum of discrepancy.

There are many implementations of this general procedure: some are simple methods that perform the analysis (curve-fit) on a very small part of the measured response function, such as in the vicinity of a single resonance on a single response curve, while others are more extensive, performing the same essential task on a much greater dataset, such as is provided by several response functions, each covering many resonances, in a single calculation. **Figure 4** illustrates examples of both types of analysis – the first case (A) showing the traditional 'circle-fit' approach to the analysis of a single resonance on a single curve, while the second case (B) shows the result of a multicurve, multimode simultaneous analysis of over 70 FRFs in a frequency range spanning several modes.

In each case, the essential process is the same: measured response data – usually in the form of a set of measured FRFs – are subjected to an extensive curve-fitting analysis in which the reference curves are based on the formula for response functions for an MDOF system with unknown parameters. By curve fitting, best estimates for these parameters are determined and these, in turn, yield the essential modal properties – natural frequencies, damping factors and mode shapes – of the model which has been proposed. Of course, it is important that the assumed model must be of a sufficiently general form and order such that it is possible to find values for its parameters that render it capable of representing the measured behavior. There can be a problem in achieving the desired result if, for example, the model is assumed to have only 10 DOFs when the measurements clearly reflect the behavior of a system with more modes than that. Likewise, conflicts will arise if the assumed model is linear while the measured data reflect the behavior of a structure which is noticeably nonlinear in its characteristic. In this respect, it is important to check that the results of the curve-fitting, modal analysis, process are not only the 'best-fit' results in a least squares sense, but also that they are good results. These two criteria are not automatically satisfied simultaneously.

For full details of the various methods for extracting modal parameters estimates for a structure from analysis of measured response functions, see **Modal analysis, experimental**, Parameter extraction methods.

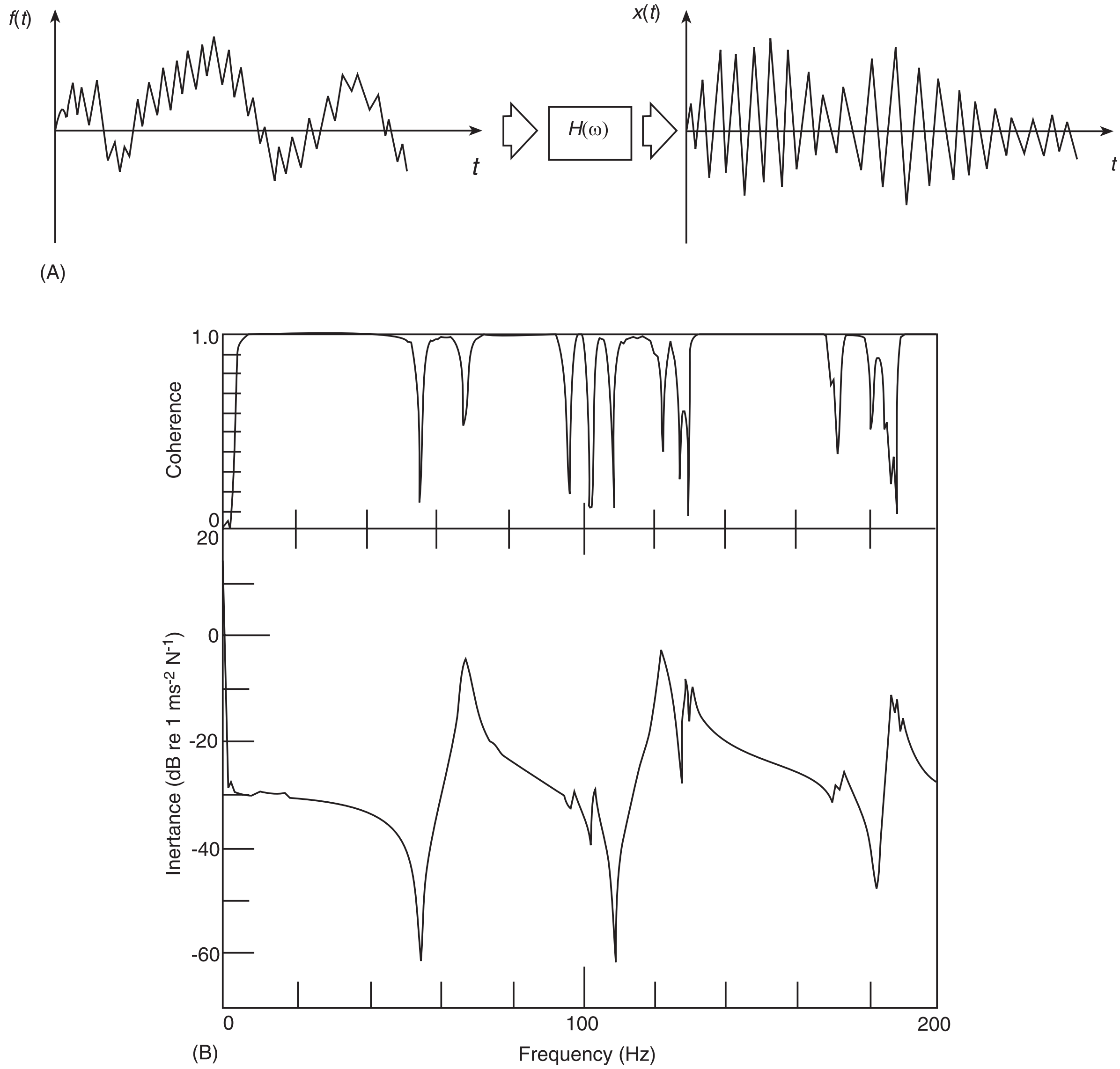


Figure 3 Typical transducer signals and frequency response function produced from modal test. (A) excitation and response signals under periodic excitation; (B) FRF produced from periodic excitation.

Construction of the Mathematical Model

The final step in the analysis of measured data is that of combining all the detailed results from the parameter extraction procedures, and using these to construct the mathematical model which is the objective of the whole exercise. In effect, once the modal parameter extraction exercise is completed, the elements for a modal model have been obtained, and a mathematical model of this type is, theoretically, available.

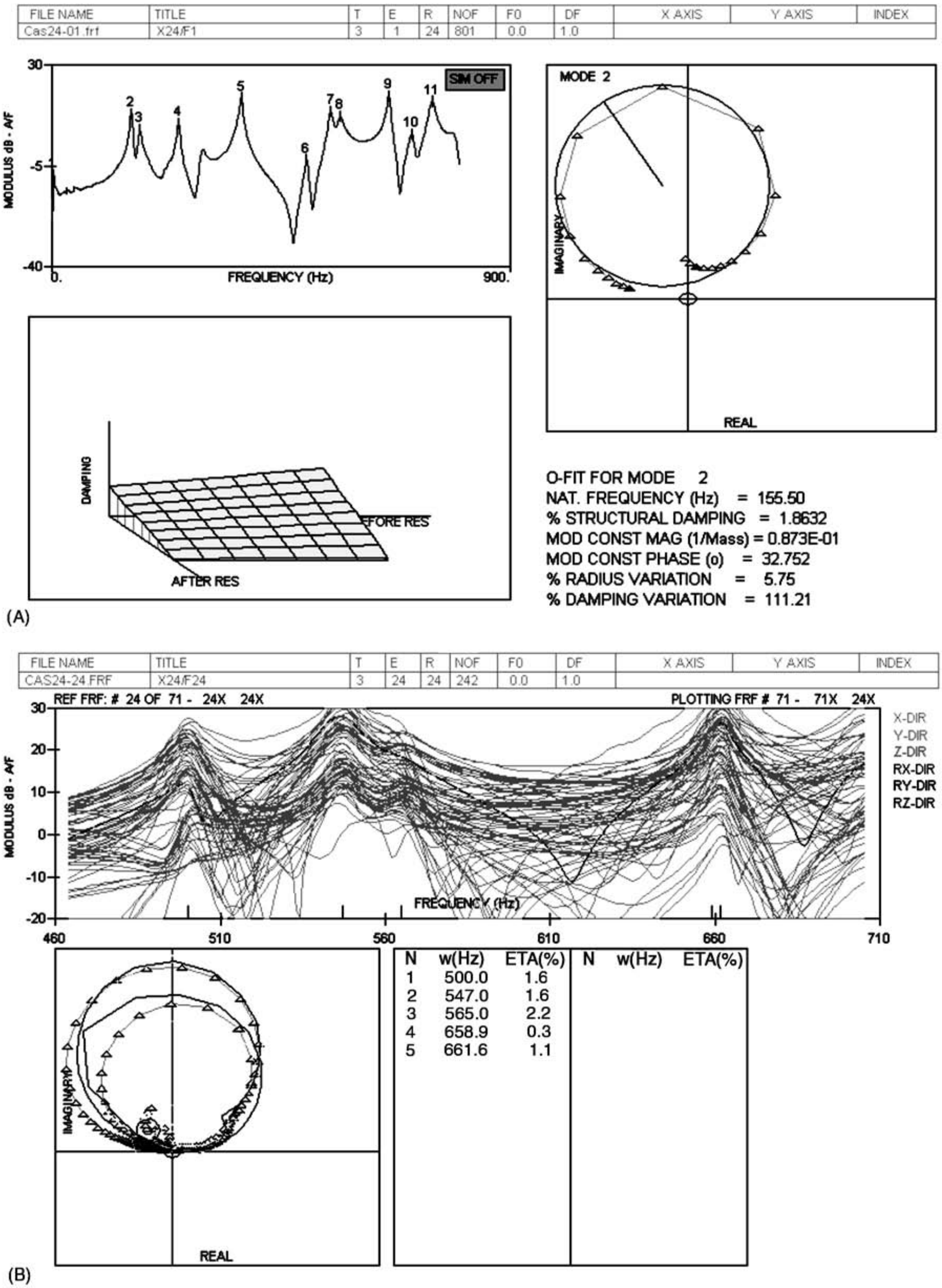


Figure 4 (See Plate 43). Example of modal analysis on measured FRF data. (A) simple, SDOF circle-fit analysis; (B) multimode; multi-DOF global analysis.

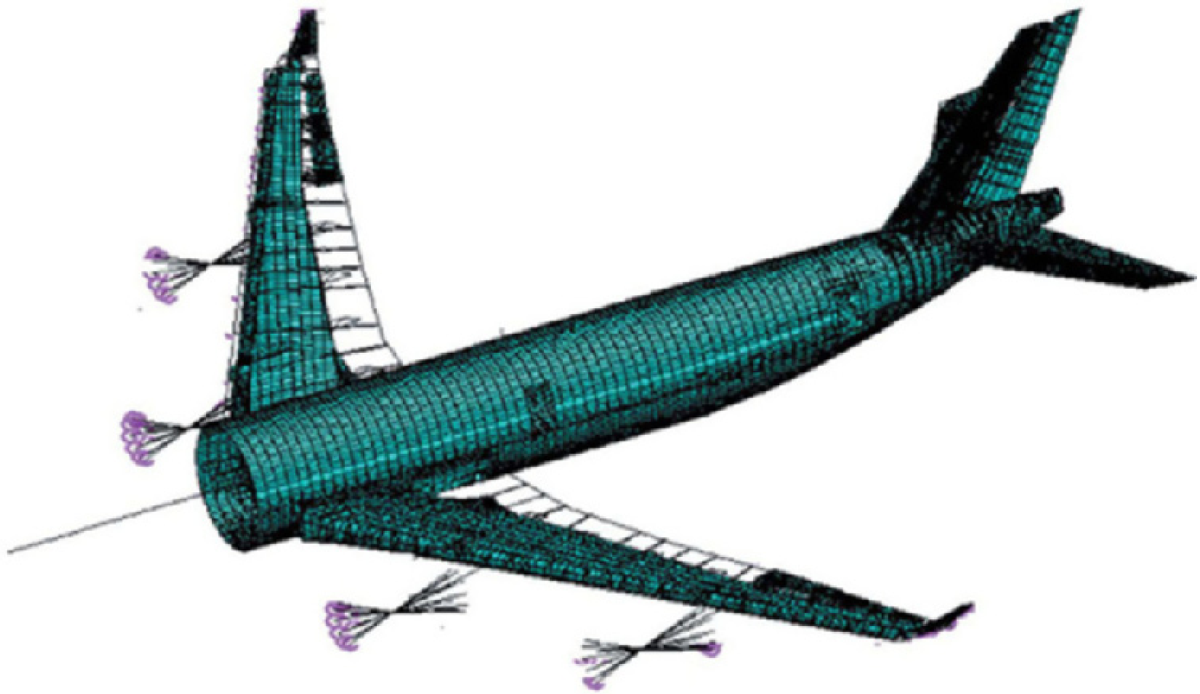


Plate 33 (above) Flutter, Active Control.
Finite element discretization of a commercial transport airplane (courtesy of the Institute of Flight Mechanics and Control).

Plate 34 (right) Flutter, Active Control.
Active decoupler pylon with adaptive piezoceramic wafer stacks.

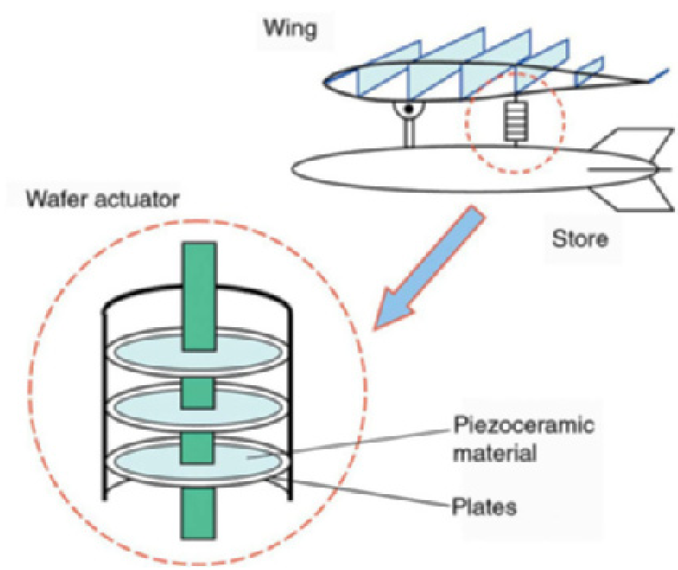
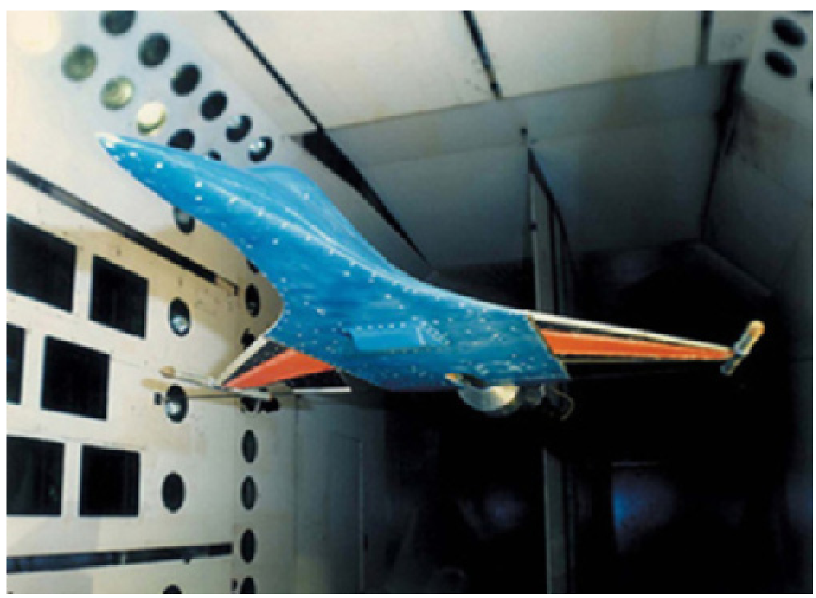


Plate 35 Flutter, Active Control.
Active flexible wing (AFW) wind-tunnel model in the NASA Langley transonic dynamics tunnel (courtesy of NASA Langley Research Center).



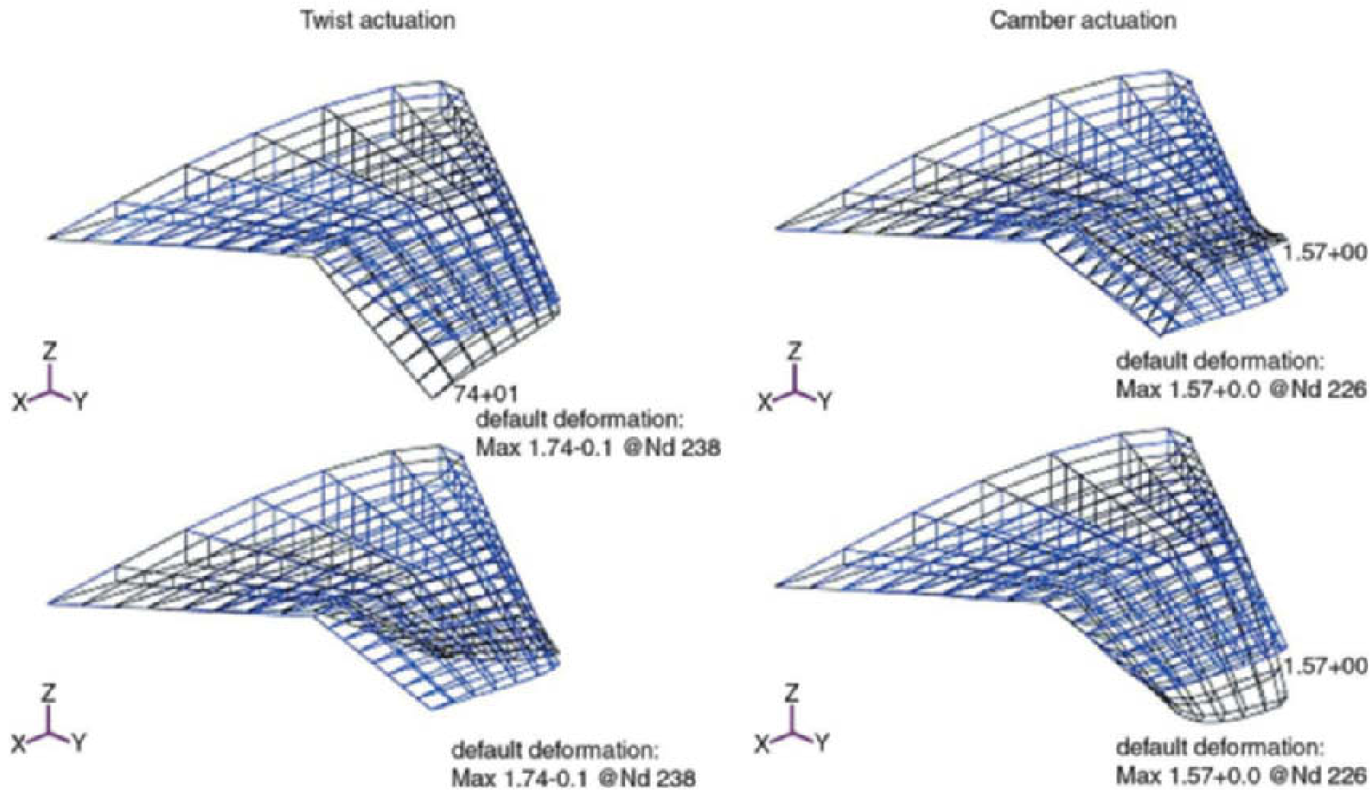


Plate 36 Flutter, Active Control. Finite element analysis of a smart UCAV wing with morphing airfoil sections.

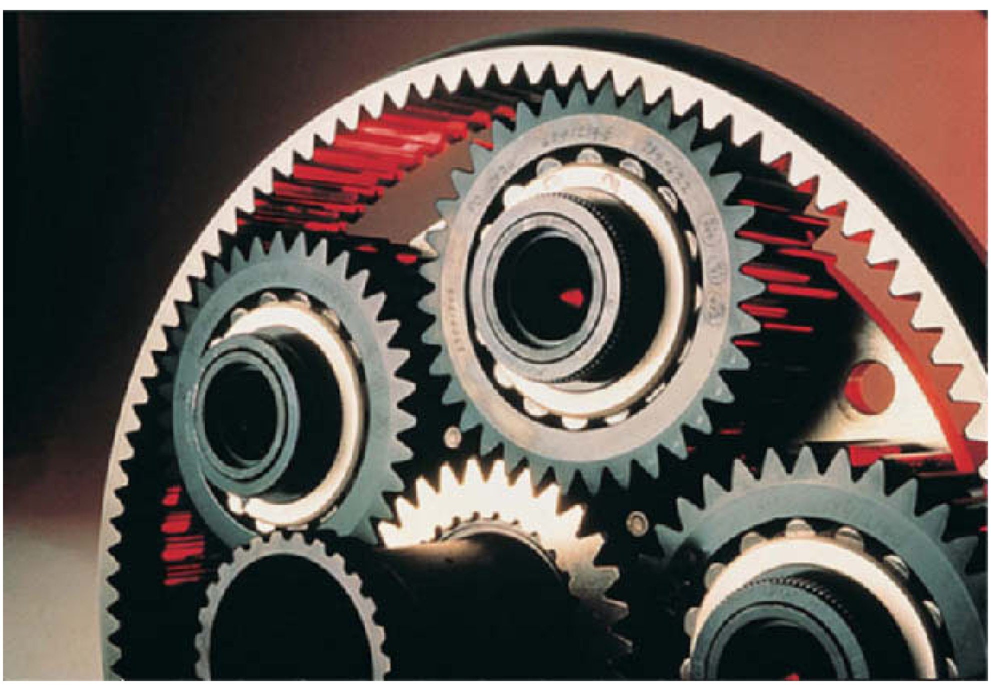


Plate 37 Gear Diagnostics. Gear from an aircraft. Part of the transmission system in an aircraft engine. This gear consists of flywheels with teeth which transmit rotating movements between shafts. (With permission from Science Photo Library).

Plate 38 Ground Transportation Systems. A scientist at General Motors using a CRAY supercomputer to produce a computer-aided design (CAD) program to simulate the aerodynamics for a new car design. (With permission from Science Photo Library).

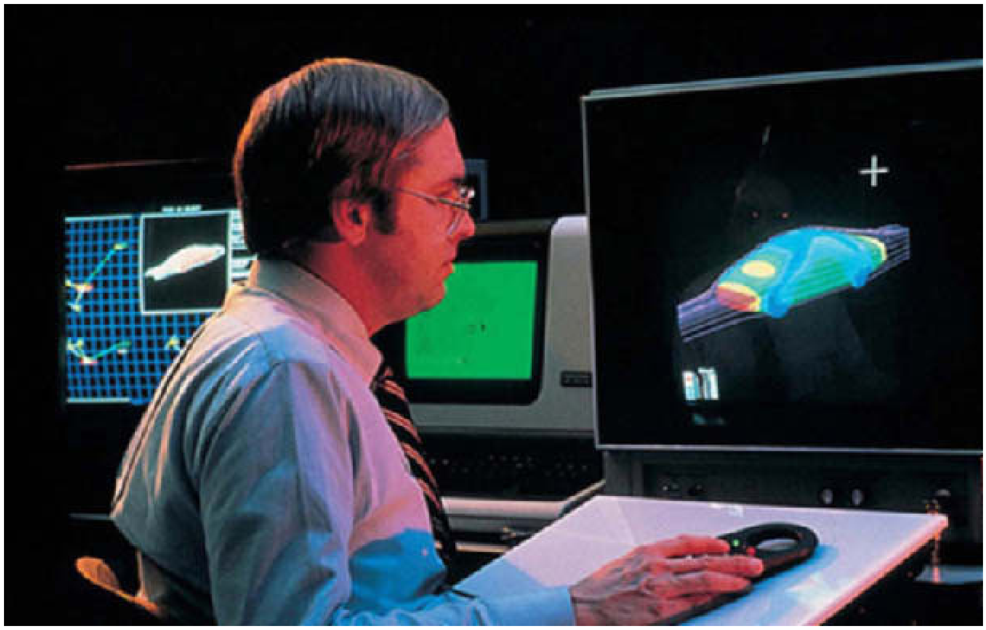


Plate 39 Helicopter Damping. Laminated metal-elastomeric bearings for isolating transmission gear box vibrations in helicopters. (Courtesy of Paulstra-Vibrachoc.)

Plate 40 Helicopter Damping. Lead-lag dampers are used to augment stability of helicopter rotor blade in-plane bending modes. (Courtesy of Paulstra-Vibrachoc.)

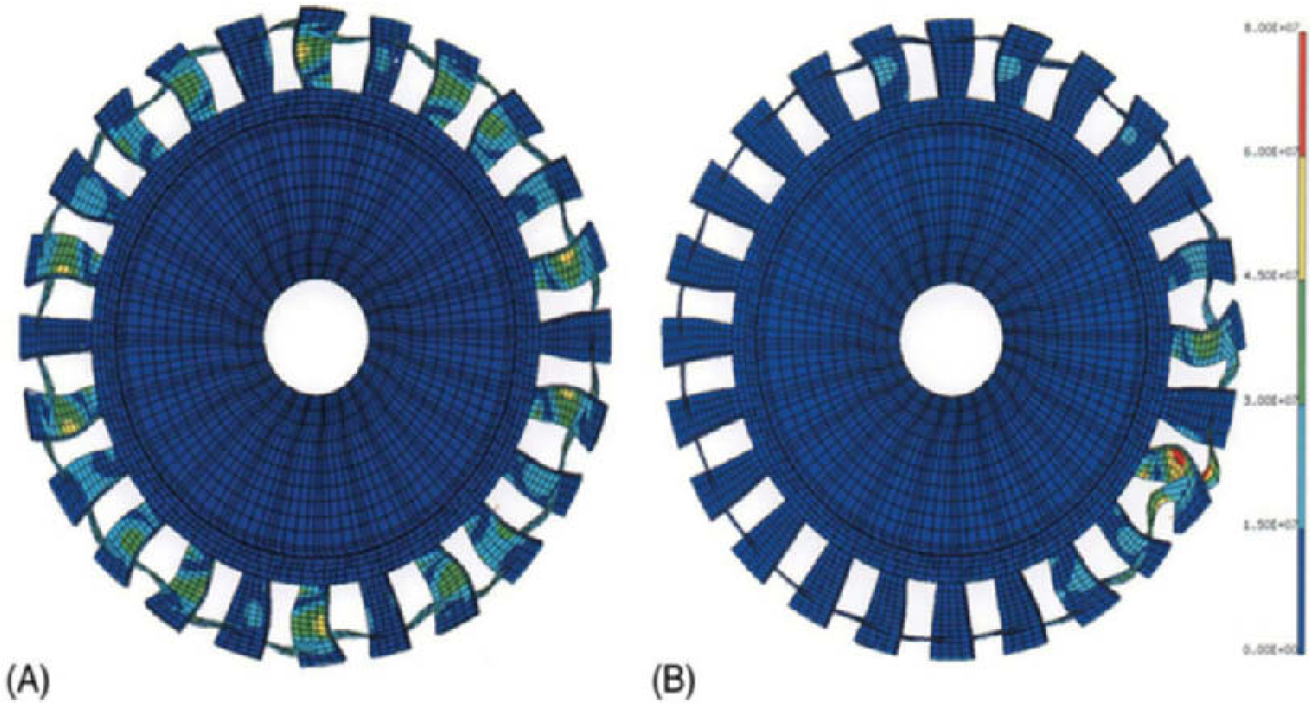


Plate 41 Localization. Forced response simulation results obtained using the finite element method. This model of a one-piece bladed disk (blisk) with continuous midspan shrouds was subject to a 'engine order 7' excitation (all blades were excited seven times per rotor revolution). (A) Extended forced response of the tuned blisk with identical blades. (B) Localized response of a blisk with a slight random blade mistuning. Observe how the largest-responding blade suffers higher vibration amplitudes and stresses than any blade of the tuned system in (A).

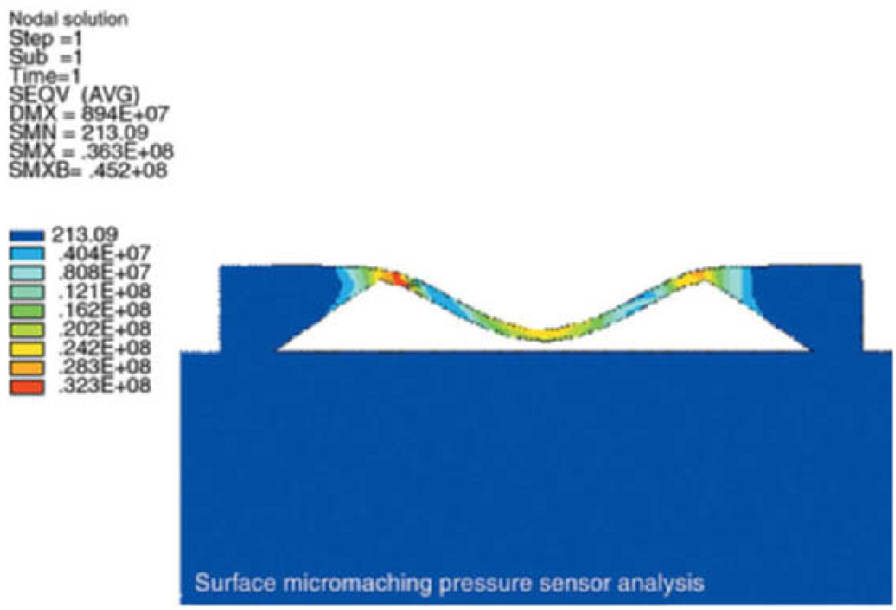


Plate 42 MEMs, General Properties. The output graphic file of an analysis of a silicon plate subjected to uniform distributed pressure loading.

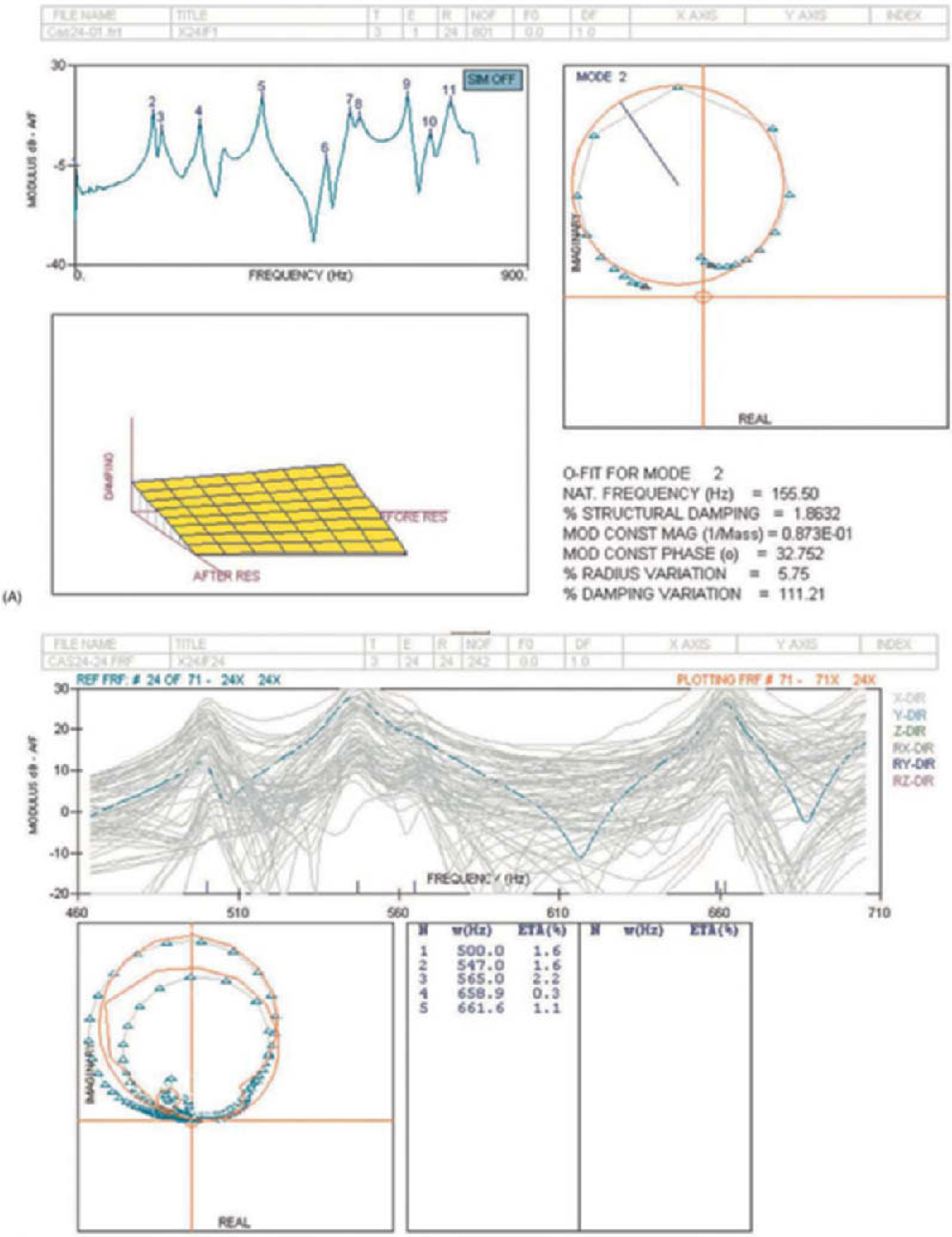


Plate 43 Modal Analysis, Experimental: Basic Principles. Example of modal analysis on measured FRF data. (A) Simple, SDOF circle-fit analysis; (B) multimode, multi-DOF global analysis.

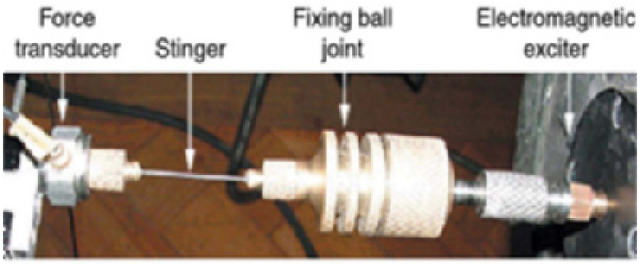


Plate 44 Modal Analysis, Experimental: Measurement Techniques. Shaker connected to test structure.

Plate 45 Modal Analysis, Experimental: Measurement Techniques. Example of multichannel FFT analyzer.

Analytical mode number	Experimental mode number 1	2	3	4	5	6	7	8	9	10
1	100	0	1	0	0	0	0	0	0	0
2	0	100	1	1	0	0	0	0	0	0
3	0	1	94	3	2	0	0	0	0	0
4	0	0	2	92	5	3	0	0	0	0
5	0	0	0	4	86	7	4	0	0	0
6	0	0	0	0	7	81	9	5	0	0
7	0	0	0	0	0	10	75	10	5	0
8	0	0	0	0	0	0	12	71	11	5
9	0	0	0	0	0	0	0	14	68	11
10	0	0	0	0	0	0	0	0	16	65

(A) *Modal assurance criterion (MAC) %*

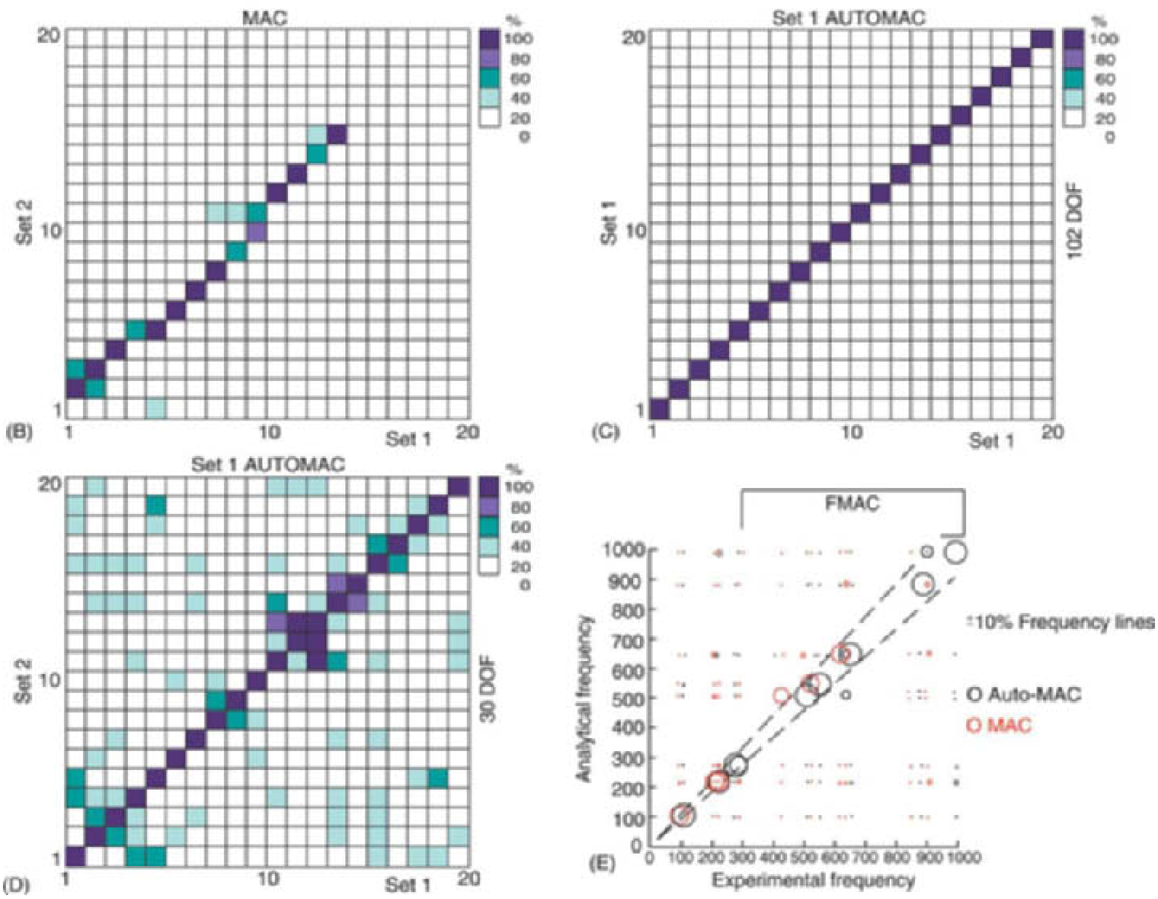
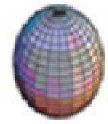


Plate 46 Modal Analysis, Experimental: Applications. Correlation of test and analysis. (A) MAC table, (B) MAC diagram, (C) AutoMAC - many DOFs, (D) AutoMAC - reduced DOFs, (E) FMAC plot.

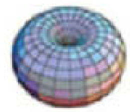
Plate 47 Noise: Noise Radiated from Elementary Sources. Spherical harmonic functions $Y_{lm}(\theta,\phi)$ and their directivities $|Y_{lm}(\theta,\phi)|$.

i =

$$Y_{00}(\theta,\phi)=\frac{1}{\sqrt{4\pi}}$$



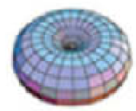
$$Y_{11}(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$



i =

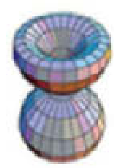
$$Y_{10}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{22}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{i2\phi}$$



i =

$$Y_{21}(\theta,\phi)=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$$



$$Y_{20}(\theta,\phi)=\sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta-\frac{1}{2})$$

Plate 48 Noise: Noise Radiated from Elementary Sources. Directivity distribution of a longitudinal and a lateral quadrupole sound radiation.

Longitudinal quadrupole

$$p(\mathbf{r},t)=\rho_o\omega Q(kd)^2\frac{e^{j\omega t-jkr}}{4\pi r}\sin\theta\cos\theta\sin\phi$$

Lateral quadrupole

$$p(\mathbf{r},t)=-j\rho_o\omega Q(kd)^2\frac{e^{j\omega t-jkr}}{4\pi r}[\frac{1}{3}-\sin^2\theta\sin^2\phi]$$

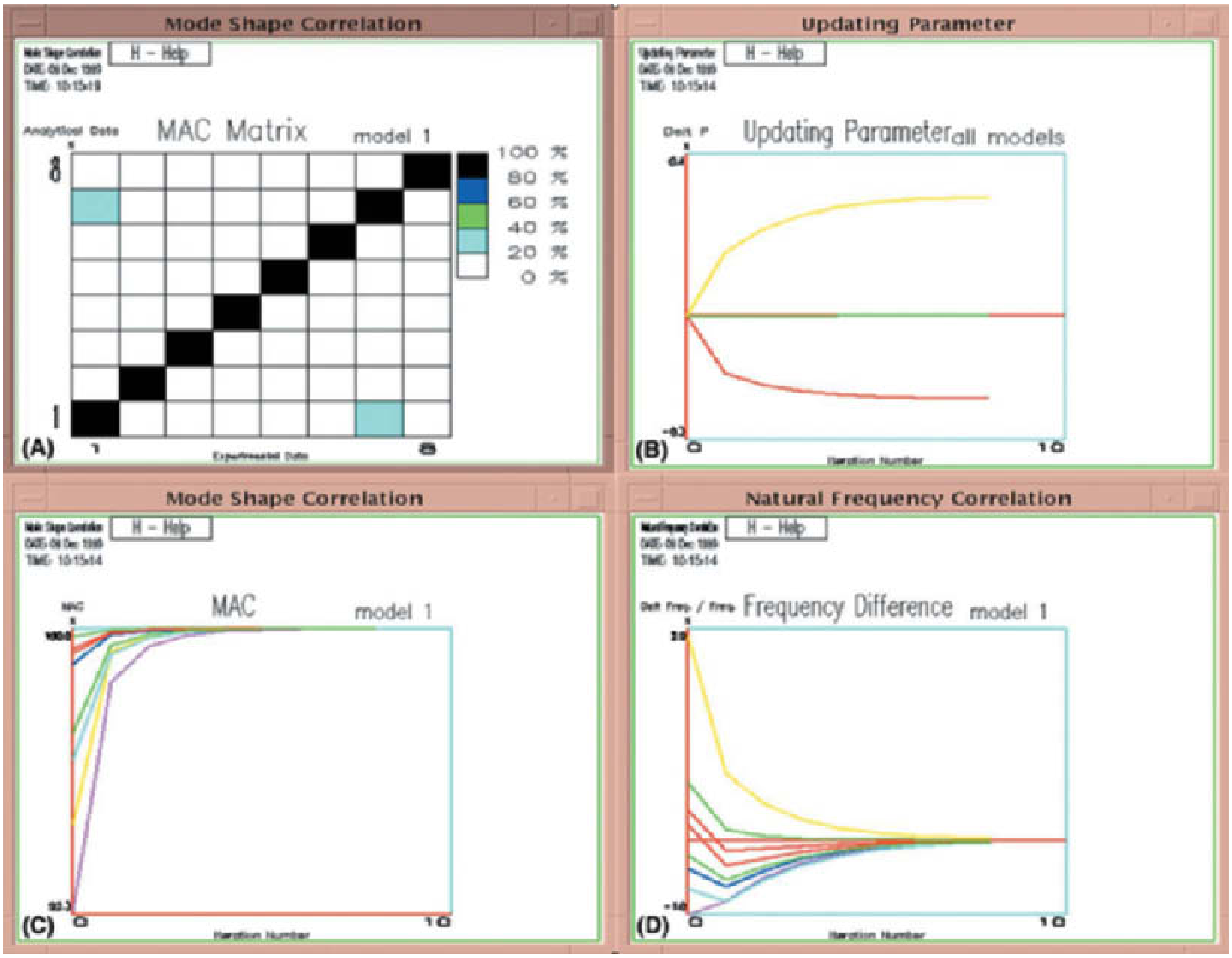
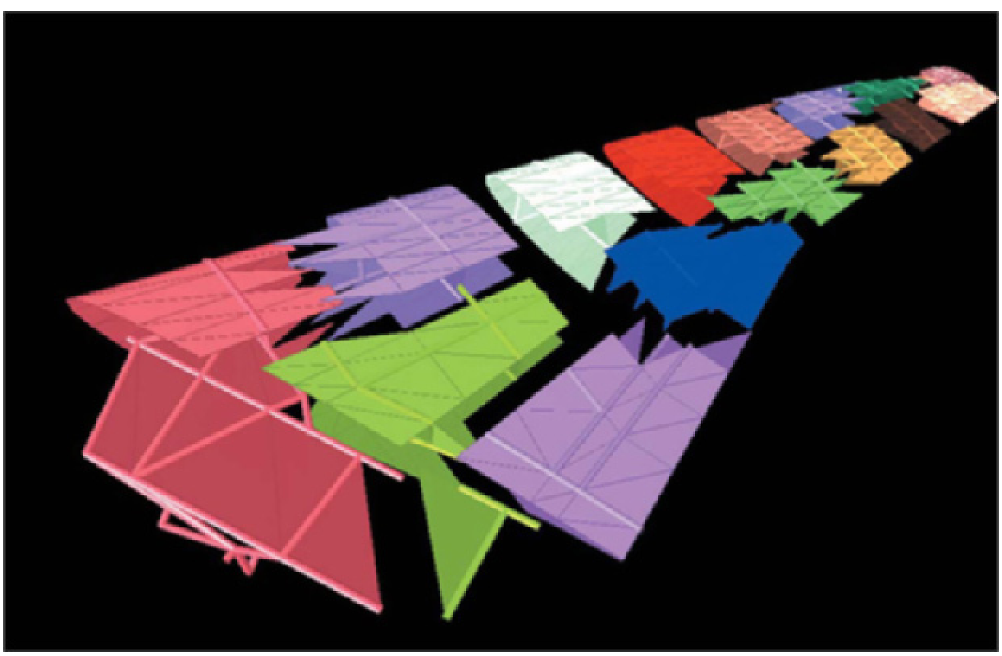


Plate 49 Modal Analysis, Experimental: Applications. Example of evolution of model updating session. (A) Current MAC matrix, (B) evolution of updated parameters, (C) evolution of MAC values for correlated model pairs, (D) evolution of natural frequency discrepancies.

Plate 50 Parallel Processing. Nonoverlapping decomposition of a finite element model. (Courtesy of the Center for Aerospace Structures, University of Colorado, Boulder, USA.)



However, there are complications that can arise in practice because of the fact that the measured data, and the resulting modal parameters that have been extracted, are not only imprecise – any measured data are liable to contain errors – but are generally incomplete. It is simply not practical to measure at all the DOFs and even less so to measure over a frequency range that captures all the modes. Thus, any model which is constructed from a practical modal test will be restricted to one which represents the behavior of only a limited number of modes and with these defined at just a small number of DOFs.

While these limitations which result from the inevitable incompleteness of the measured data are not particularly problematic when considering the modes of the structure, they can become a problem when seeking to convert the measured modal model into a spatial (mass-stiffness) model or to a response model. The equations given above which define the interrelationships between these three types of model can be seen to become inapplicable if the description in the original model is not complete – even if the data which are included are fully accurate. This presents something of a problem for some applications as several of the more popular ones really require the derived model to be in spatial or response form (see **Modal analysis, experimental**, Applications). For a full discussion of the issues which arise in constructing a suitable model from the measured and analyzed vibration data, see **Modal analysis, experimental**: Construction of models from tests.

See Plate 43.

See also: **Modal analysis, experimental**, Applications; **Modal analysis, experimental**, Construction of models from tests; **Modal analysis, experimental**, Measurement techniques; **Modal analysis, experimental**, Parameter extraction methods.

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Measurement techniques

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Introduction

The experimental determination of natural frequencies, mode shapes, and damping ratios is called experimental modal analysis and is based on vibration measurements that fall within the general designation of modal testing. The objective of this form of vibration testing is to acquire sets of frequency response functions (FRFs) that are sufficiently accurate and extensive, in both the frequency and spatial domains, to enable analysis and extraction of the dynamic properties for all the required modes of vibration of the structure. Prior knowledge of areas such as vibration analysis, instrumentation, signal processing, and modal identification, to state just a few, is required to understand modal testing.

The basic aim of modal testing is to obtain FRFs relating output vibration responses at a number of coordinates of interest, usually under the form of accelerations (or velocities, or displacements), to input vibration excitations, usually under the form of driving forces, applied at a given coordinate.

Basic Measurement System

In order to perform modal testing a number of hardware components must be available. These components may be interfaced with a host computer allowing for coordination of the operation of the overall system and enhancing the data-processing capabilities, if adequate software is available. The hardware components are schematically represented in **Figure 1**, which shows a typical set-up for a measurement system. Basically, there are three main measurement mechanisms:

- the excitation mechanism
- the sensing mechanism
- the data acquisition and processing mechanism

Adequate selection of an experimental modal analysis system is not an easy task and requires good understanding of the underlying theory.

The Excitation Mechanism

The excitation mechanism is constituted by a system which provides the input motion to the structure under analysis, generally under the form of a driving force $f(t)$ applied at a given coordinate. There are many variants for this system, their choice depending on several factors such as the desired input, accessibility, and physical properties of the test structure. Basically there are fixed exciters and non-fixed exciters.

The fixed exciter, also known as shaker, is usually an electromagnetic or electrohydraulic vibrator, driven by a power amplifier (**Figure 2**). The excitation signals, in these cases, are generated by a signal generator and can be chosen from a variety of different possibilities (stepped-sine, swept sine, impulse, random, etc.), to match the requirements of the structure under test.

This type of excitation mechanism may be easily controlled both in frequency and amplitude and therefore offers the best overall accuracy. However, it also has some disadvantages, such as the need for the exciter to be connected to the test structure. Despite the use of connecting devices (named pushrods, drive rods, or stingers) designed to reduce the attachment consequences, there are always some constraining effects and mass loading of the structure.

Conventional electromagnetic or electrohydraulic exciters vary in size and their choice depends on the structure under test. The main characteristics to take into consideration are force level, displacement level, and frequency range. The objective is for the exciter to provide inputs large enough to result in easily measured responses. The applied excitation force is commonly measured by means of a load cell (known as force transducer) which is located at the end of the stinger and is rigidly connected to the test structure (**Figure 3**).

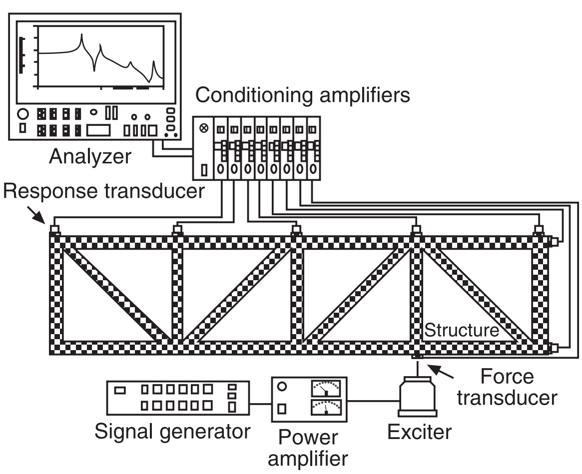

Figure 1 Typical measurement set-up.

Figure 2 Example of electromagnetic exciter and matching power amplifier.

The nonfixed exciter, known as an impulse or impact hammer, is a very popular alternative to fixed systems. It consists of a hammer with a force transducer attached to its head (**Figure 4**). The most important advantage of these excitation systems is that they do not need a signal generator and a power amplifier and nothing is attached to the structure. Hence, the excitation system does not affect the dynamics of the test structure.

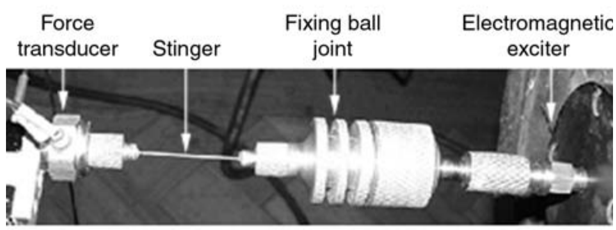

Figure 3 (see Plate 44) Shaker connected to test structure.

Figure 4 Impact hammer with force transducer and soft head.

The range of frequencies covered by a hammer depends on the hammer mass and on how hard its impacting head is (note that the mass and stiffness of the impacted structure also contribute to define this range). Furthermore, the mass and therefore the size of the hammer, together with the velocity of the impact, dictate the amplitude of the impact force. Impact hammers concentrate the total input excitation energy in a very short period of time, producing an almost flat force spectrum over a wide frequency range. Problems may arise due to local deformations and nonlinearities.

Preloading a structure and suddenly releasing it, exciting with acoustic energy (using a loudspeaker) or with a magnetic device, are other alternative nonfixed excitation possibilities. The problem with most of these cases (except in the preload/release alternative) is that the exciting force cannot be measured.

The Sensing Mechanism

The sensing mechanism is, basically, constituted by sensing devices known as transducers. There is a large variety of such devices: the most commonly used in experimental modal analysis are the piezoelectric transducers either for measuring force excitation (force transducers), as exemplified in **Figure 5**, or for measuring acceleration response (accelerometers), as exemplified in **Figure 6**. The transducers generate electrical signals that are proportional to the physical parameters one wants to measure.

Most of the time, the electrical signals generated by the transducers are not amenable to direct measurement and processing. Such a problem, which is usually related to the signals being very weak and to electric impedance mismatch, is solved by the use of conditioning amplifiers. These devices are usually considered as part of the transducers and therefore of the sensing mechanism (some transducers actually incorporate the basic conditioning electronics).

The choice of transducers is based on maximum force or maximum motion response, frequency range (which depends on the load), size (and therefore mass), and sensitivity.

Most transducers used in modal analysis have to be fixed to the structure under test and, thus, they may affect the measured dynamic response. This is especially true in the case of small lightweight structures. A common alternative is to use a contactless motion transducer, such as a laser vibrometer. This is a velocity transducer based on detection of the Doppler frequency shift of a laser beam light scattered from the moving surface.

Data Acquisition and Processing Mechanism

The basic objective of the data acquisition and processing mechanism is to measure the signals developed by the sensing mechanism, and to ascertain the magnitudes and phases of the excitation forces and responses. There are very sophisticated devices for this purpose, called analyzers, that incorporate many functions and even include the signal generation component. The most common analyzers are based on the fast Fourier transform (FFT) algorithm and provide direct measurement of the FRFs. They are known as spectrum analyzers or FFT analyzers. Basically, they convert the analog time domain signals generated by the transducers into digital-frequency domain information that can be subsequently processed with digital computers. Multichannel FFT analyzers (**Figure 7**) are a normal component of any modal analysis laboratory.

Due to the rapid development of computers, there is nowadays a tendency for the computer to replace the analyzer, provided it incorporates adequate software and a data acquisition board. More sophisticated systems are based on personal computers (PCs) or workstations with an acquisition front-end module. These solutions have the additional advantage of allowing for storage and data processing without having to go through the intermediate steps of data transfer. If based on a portable computer (lap-top), these systems may easily be carried to the test site.

The PC-based systems are usually cheap. However, one must be careful in choosing the data acquisition

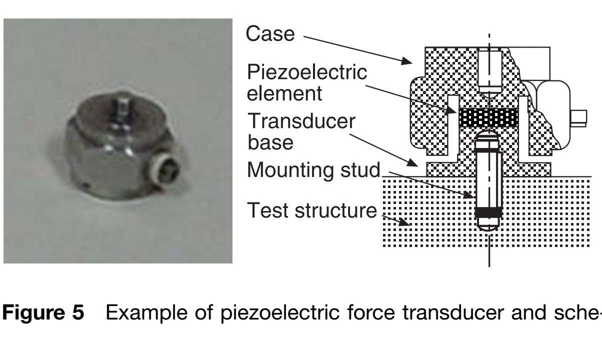

Figure 5 Example of piezoelectric force transducer and sche-

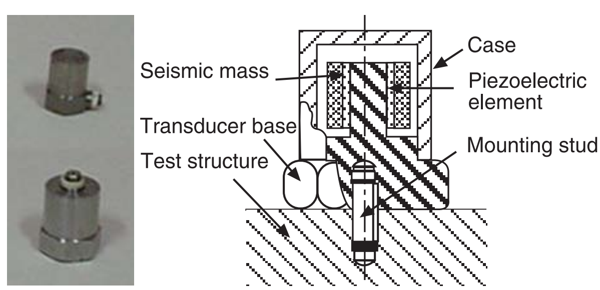

Figure 6 Example of piezoelectric accelerometers and schematic representation of their cross-section.

Figure 7 Example of multichannel FFT analyzer.

plug-in boards: Sampling rate, analog-to-digital conversion (ADC) accuracy, frequency range, filter characteristics, and zoom capabilities are some of the characteristics to take into consideration.

Understanding the principles behind signal acquisition and processing is very important for anyone involved with the use of modal analysis test equipment. The validity and accuracy of the experimental results may strongly depend on the knowledge and experience of the equipment user.

Excitation Methods and Measurement

There are various excitation signals that are commonly used in experimental modal analysis. They may be classified as single-frequency and broadband excitation signals, and each technique incorporates several different methods. Although each method has specific advantages and disadvantages, selection of a testing technique is often based (unfortunately) on the type of equipment and expertise available rather than on its suitability for a particular job.

Single-Frequency Excitation

Stepped sine and swept sine are the two types of single-frequency excitation signals commonly used.

In the stepped sine testing technique, a shaker is used to excite the structure sinusoidally at a single, precisely controlled frequency. The structure is allowed to settle under this excitation, to remove any transient effects. Steady state measurements are then made of the magnitude and phase relationship between the input force and the response at the precise excitation frequency for any desired response location. Division of the response by the force input gives the value of the FRF at that particular frequency. The excitation frequency is then changed by a small increment and the measurement process is repeated once the structure has settled at the new frequency. Thus, the FRF can be constructed for any frequency range.

One of the advantages of the stepped-sine testing technique is the large signal-to-noise ratio for all the force and response measurements. Furthermore, the input ranges for the force and response signals can be adjusted automatically for each excitation frequency point, and this allows the best possible use of the measuring equipment. This technique is especially adequate to investigate nonlinear behavior.

In the swept-sine testing technique, a shaker is used to excite the structure with a sine signal whose frequency varies slowly and continuously along the test frequency range. Sets of simultaneous measurements of the excitation signal and response signals are then taken at a given rate. The frequency of the sine signal allows a low rate of change, assuming that each set of measurements correspond to virtually steady state characteristics, i.e., that the technique sinusoidally excites the structure for each set of measurements at virtually one single frequency.

Though similar, swept sine techniques are faster than stepped sine techniques. However, since there is a continuous rate of frequency shift, accuracy is affected.

Broadband Excitation

In broadband testing, the structure is excited with a signal containing energy over a wide range of frequencies simultaneously. The time domain force and response signals are filtered, digitized and then passed through a Fourier analysis process to transform the time domain information to frequency domain spectra. By appropriate combination of the force and response spectra, the required FRFs for the structure can be derived.

Broadband excitation techniques may be classified as nonperiodic (purely random), periodic (pseudorandom, periodic random, and periodic chirp) and transient (burst random, burst chirp, and impact). Random, periodic random and impact are probably the most commonly used types of broadband excitation signals.

The pure random excitation signal is a nonperiodic stochastic signal with a Gaussian probability distribution. For a linear system, a Gaussian input will produce a Gaussian output. Since the peak-to-root mean square ratio of the output is relatively small, the random-type excitation does not cause undue problems with excitation of nonlinear structures and easily averages out noncoherent noise. However, because the force and response signals are random, a weighting function (e.g. Hanning window) must be applied to these signals before the discrete Fourier transform (DFT) is performed, otherwise the mathematical assumptions of periodicity in the measurement time frame are invalid.

The force and response signals from the transducers contain energy at all frequencies. Because the measurement time frame is of a finite length, the Fourier analysis produces spectra that are discrete, rather than continuous, functions. Therefore, there is a 'spreading' of energy from the continuous spectra into adjacent spectral lines producing a phenomenon known as 'leakage'. The use of adequate weighting functions (windows) reduces the above-mentioned 'spreading'.

Periodic random excitation is a special form of periodic excitation, similar to pseudo-random, that has several benefits for signal processing. Although called 'random' it is not random in the true sense of the word. Within a time period equal to the analysis time frame, the signal is random, but the same signal is then repeated continuously. Usually, the signals are generated inside the data acquisition equipment in the frequency domain. The spectrum of a periodic random signal consists of discrete frequencies at integer multiples of the frequency resolution used by the DFT. Once the measurement frequency range and number of frequency lines have been selected, a flat excitation frequency spectrum may be generated by setting the magnitude of all spectral lines to the same value. The phases of the spectral components are then randomized. By use of the inverse Fourier transform (IFT), this frequency domain spectrum is transformed into the time domain to produce a random-like excitation signal in the analysis time frame. As a consequence of generating the signal in this way, it is exactly periodic in the analysis time frame, and both the force and response signals will be periodic in the analysis time frame also.

Provided that the excitation is applied to the structure more than once, so that any start-up transients have had sufficient time to decay, there is no requirement for any window function to be applied to this type of data. These signals only contain energy at the precise spectral component frequencies of the analysis and, therefore, there are no leakage problems when periodic random excitation signals are used. Furthermore, the signal-to-noise ratios for the measured force and response signals are much better than they would be with pure random excitation because there is no superfluous energy contained in the excitation.

The impact excitation is based on an input transient deterministic signal consisting of a pulse lasting for only a very small part of the sampling period. The shape, width, and amplitude of the pulse determine the frequency content of the force spectrum.

Impact excitation is usually obtained with impact hammers. The shape and amplitude of the impact signal determine the level of the force spectrum. The base-band frequency span is controlled by the width of the impact signal. The maximum frequency is inversely proportional to the width of the pulse signal.

Multipoint Testing

In tests on large complicated structures, with numerous joints and nonlinearities, the vibration energy is quickly dissipated within the structure. This may indicate that the use of multipoint testing is preferable to a series of single-point tests. By the use of multipoint excitation, the energy can be fed into the structure more uniformly than with single-point excitation and the response amplitudes at various locations can be kept much closer to those found in operation. Energy is supplied to the structure by several shakers, and so smaller and cheaper shakers can be used than would be necessary for single-point testing. Furthermore, the effects of nonlinearities, which may occur with excessive single-point forcing, can be substantially reduced. Since all the shakers are connected to the structure before the start of the test, systematic errors, resulting from repositioning the shaker during a series of single-point tests, do not occur. The influence of the shakers on the structure is not removed; it just remains constant throughout the whole of the test program. Simultaneous measurement of multiple columns of the FRF matrix means that the overall test time is shortened. There is less opportunity for structural changes (with time, temperature, or humidity) to affect the measured results.

Multipoint testing may be expensive in terms of suitable test control hardware and software that are required. Also, the process for extracting the FRFs is more complicated than for a single-input test.

The most commonly-used multiple shaker technique is known as multipoint random (MPR). For all multipoint excitation techniques, more sophisticated computer programs are necessary to extract the standard FRFs from the measured data.

To comply with the mathematical assumptions made in the analysis, it is important that the excitation inputs to the structure are purely random and uncorrelated. Although it is possible to ensure that the excitation signals driving the shakers are uncorrelated, it does not necessarily follow that the excitation force signals are uncorrelated. At the resonance frequencies, in particular, it is found that the motion of the structure tends to correlate the multiple forcing inputs. This can lead to degradation in the quality of the derived FRFs at the resonance frequencies.

Special types of multipoint excitation methods are known as 'phase resonance methods' (also known as 'tuned-sinusoidal' or 'force appropriation methods') and these rely on the ability to excite a single mode of vibration by the use of multiple shakers with independently variable force levels. All the shakers produce sinusoidal excitations at the same frequency and are either in-phase or out-of-phase with a reference source.

The excitation set-up in the 'normal mode' technique effectively cancels the damping in the structure the exciting forces are distributed such that each energy sink is canceled by a corresponding energy source and single real modes can be excited. In this condition of normal-mode vibration, the excitation frequency is the undamped natural frequency of the mode. The response at all the points on the structure is in quadrature with the excitation forces, and the structural responses relate directly to the mode shape vector.

The previous method provides the ability to measure real normal modes (for direct comparison with finite element results) and the ability to investigate nonlinear behavior. The main difficulties are the selection of excitation locations, the tuning of the force pattern, and the choice of the excitation frequency. The complete process has to be repeated for each different mode and consequently the testing time can be lengthy.

Calibration

The values measured by the test equipment represent electrical voltages and therefore it is necessary to obtain a calibration factor which translates these values into units of acceleration and force.

Though transducer manufacturers generally provide reliable calibration information, the use of their quoted sensitivities may not be accurate enough since they can change with time and environmental conditions. In addition, the transducers may have suffered some kind of damage due to rough handling or other extreme conditions and, although still working, may have lost their response linearity. Finally, the remaining units in the measurement chain (amplifiers, filters, signal conditioners, cable lengths, etc.) may change, albeit slightly, the overall sensitivity. It is therefore good practice to recalibrate the transducers before performing a test, preferably using the same measuring set-up that will be used in the test program.

Various transducer calibration techniques are available to the test engineer. The simplest and most common calibration procedure is the classical back-to-back method that compares the accelerometer to be calibrated with a reference accelerometer. This entails keeping a special reference transducer that offers a high level of linearity and stability. Another simple calibration procedure is based on the use of small hand-held calibrators that most manufacturers commercialize. In this case, the calibration is performed at one frequency only and, therefore, a flat frequency response of the transducer over the frequency band of interest is assumed.

When measuring FRF data, one is concerned with the motion/force ratio and not with the individual values of any of these quantities. This fact allows the use of a simple and straightforward technique which provides an accurate calibration of the transducers, including the influence of the remaining units of the measurement chain. The technique requires only the use of a simple rigid structure, such as a steel block, together with the equipment that is going to be used for the accelerance measurements.

For each accelerometer to be used, it is necessary to make a calibration test involving simultaneously the accelerometer and the force transducer. Applying a time-varying force to a solid block of known mass (which can be accurately measured), measuring the corresponding acceleration response, and computing the accelerance through a specified frequency range, one obtains a value in units of volt/volt which corresponds to $1/m$, where m is the mass of the block (which may include the added transducers masses). Thus, the measured accelerance will be a constant value proportional to the block mass, within a frequency range for which the block behaves as a rigid body. This calibration technique does not use the individual transducers sensitivity values and must be performed for each pair of accelerometers/force transducers.

Support Conditions

The support conditions of the structure under test are an important part of the test set-up. They must be well defined and experimentally repeatable if the results of the measurements are to reflect the properties of the structure without undue influence from the support. For test of components *in situ*, exact definition of the boundary conditions may be problematic but, nevertheless, tests should be considered to prove the repeatability of the installation. When the aim of the test is to determine the dynamic characteristics of a system under operating conditions, the test boundary conditions should be as close as possible to the operating conditions.

When testing in the laboratory, the most frequently used boundary conditions are grounded or free conditions. It is almost impossible for either of these two conditions to be achieved in practice. The most difficult to achieve is the grounded condition as the structure will always have some movement at the grounding point (usually rotation). Free conditions are easier to achieve, although there will always be some small restraints.

For a structure to be really free, it should be suspended (floating) in the air, free in space with no holding points whatsoever. Such a situation is commonly designated as 'free–free', 'freely supported' or 'ungrounded' and is clearly impossible. However, simulation of free–free conditions can be closely approximated. It suffices to suspend or support the structure using very flexible (also designated as soft) springs so that the resonance frequencies of the mass of the structure on the stiffness of the supports or suspension devices are very low and far away from the frequency range of interest.

Pretest Decisions and Checks

Performing modal analysis tests is a time-consuming task involving qualified operators and delicate equipment. As a consequence, it is an expensive task. Thus, great care should be taken in preparing the test set-ups and in performing a number of checks prior to starting the real tests. All available previous knowledge about the system under test should be taken into consideration. In addition, it may be important to have accurate information about the aim of the tests, the required data, and the required accuracy of the measured data.

Parameters that must be carefully defined include, among others, the frequency range of interest, the selection of the transducer (response and excitation) locations, the selection of the suspension locations (when adequate), and the type of excitation to be used.

A large number of transducers on a structure sometimes leads to mistakes related to the correct identification of their location and values of the calibration factors when introducing the information into the analyzing system. Such mistakes may completely destroy the validity of the final measured data unless discovered in time and rectified.

Common problems that may be overlooked are related to broken leads, badly connected cables, badly attached transducers, and even pieces of equipment that are not powered (signals do show even in these cases). Careful checks will avoid most of these problems. An oscilloscope may be a valuable piece of hardware for this purpose.

Another type of verification, which is very important, is related to the efficiency of the inputs. In case of random-type excitation, the autopower spectrum of the input should be observed in order to verify if it maintains about the same level over the test frequency range, though it may be noisy. In case of impact excitation, the input autopower spectrum should be clean and flat up to the maximum frequency of interest.

Validation of Measurements

One of the problems facing the test engineer is an inability to ascertain the quality of the measured data. Throughout the complete modal test, checks should be made to assess this quality. There are several techniques in general use that can provide an indication of the quality of the measured data, e.g. repeatability, reciprocity, and coherence.

Almost always, repeatability and reciprocity checks are done by comparing sets of FRF curves to see if there are any major differences. The comparisons are made significantly easier if difference function curves (ΔFRFs) are plotted for the sets of data.

Repeatability checks are performed by repeating some measurements and comparing the results with previously measured curves. These checks assess the stability of the structural characteristics over a period of time. It is usually assumed that the structure does not change with time or as a result of the excitation itself but there are a number of practical effects, such as bolt slackening, fretting, change of temperature and humidity, etc., that can alter the characteristics of a structure.

Reciprocity checks are based on Maxwell s rule of reciprocity. The FRF matrix is symmetric and this property can be used as a check on the quality of the measured data. In fact, for a linear conservative system, the FRF measured for a force at location j and a response at location i should correspond directly to the FRF measured for a force at location i and response at location j.

Where a multiple single-input test (various separate single-input tests with the shaker located at a different position for each test). strategy is used, the reciprocity check can give an indication of shaker and accelerometer loading effects on the structure. The positions of the shaker and accelerometer are reversed in multiple single-input reciprocity checks. If the shaker and accelerometer have negligible effect on the structure, then there should be good reciprocity. If the shaker and accelerometer have a significant loading effect on the structure, then the effects in the two configurations will be different and the reciprocity check will reveal any differences between the FRFs.

Nowadays, almost all spectrum analyzers incorporate the calculation of the coherence, which is nothing but a correlation coefficient that measures the degree of consistency of all averages of the FRF evaluated by the analyzer. A coherence equal to 1 indicates that each average is exactly the same. Low coherence values indicate a significant variance on the averages and, therefore, poor data quality. Usually, low-frequency regions and regions close to resonances and antiresonances yield low coherence values. The reason for this is poor performance of many transducers at low frequencies and low signal-to-noise ratios close to resonances and antiresonances.

See Plates 44, 45.

See also: **Modal analysis, experimental**, Applications; **Modal analysis, experimental**, Basic principles; **Modal analysis, experimental**, Construction of models from tests; **Modal analysis, experimental**, Parameter extraction methods.

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Parameter extraction methods

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Introduction

The measurement of frequency response functions (FRFs; see **Modal analysis, experimental**, Basic principles), impulse response functions (IRFs), or simply free decay responses (either of these constitute the response model) may not be enough for the envisaged subsequent applications in modal analysis. Most of the time it is necessary to build a modal model and sometimes a spatial model (see **Modal analysis, experimental**, Construction of models from tests). The modal model comprises the so-called modal parameters, which are the natural frequencies, damping ratios, and mode shapes (amplitudes and phases). The natural frequencies and damping ratios are charateristics of the structure with its boundary constraints and are therefore related to the poles of the transfer function, whereas the modal constants, amplitudes and phases (coming from the mode shapes or vice versa) depend on the locations where the responses and forces are taken and are related to the residues of the transfer function.

Methods that estimate the modal parameters, or the poles and residues to which they are closely connected, from measured responses are called indirect methods (of system identification). They allow for the construction of the modal model. The spatial model is composed of the mass, stiffness, and damping matrices of the system. Methods that evaluate those matrices are called direct methods, as they go directly from the response model to the spatial model without computing the modal parameters.

All methods that allow for the identification of the dynamic properties (or parameters) from measurements made on a real structure or machine are called Parameter extraction methods and are the subject of this article.

Classification of Methods

There are many different methods to extract the modal parameters from measurements made on a structure. The main division between them is in the domain in which the experimental responses are treated. With respect to this, there are time-domain and frequency-domain methods. The first methods to appear, around 60 years ago, were very simple and fundamentally in the frequency domain. The main objective was to determine the resonance frequencies. At the time, modal testing was known as resonance testing.

In both time and frequency domains, there are indirect and direct methods. Within a given frequency range where measurements have been taken, there are usually various resonances. Most methods allow for the extraction of the dynamic properties taking into account all the resonances simultaneously. They are called multi-degree-of-freedom (MDOF) methods. However, there are some indirect methods in the frequency domain that can estimate the modal parameters by making a separate and progressive analysis around each resonance until the whole frequency range is covered. These are called single-degree-of-freedom (SDOF) methods.

A last classification has to do with the number of time histories (or IRFs) or FRFs that are processed at the same time. If only one FRF relating one input to one output is being analyzed, the method is called a single-input single-output one (SISO); if a method simultaneously takes data from various FRFs representing different responses due to a single input location, it is called single-input multi-output (SIMO);

when FRFs relating various outputs and inputs are considered in the analysis, the method is called multi-input multi-output (MIMO). **Figure 1** illustrates the general classification of methods.

Time-domain Methods

Within the indirect methods, probably the best known ones in the time domain are the complex exponential method, which is an SISO method, and its extended versions, the least-squares complex exponential (SIMO) and polyreference complex exponential (MIMO) methods. Also very well known are the Ibrahim time domain (SIMO) and the eigensystem realization algorithm (MIMO) methods.

Within the category of direct methods the main techniques are the autoregressive moving average (SIMO) and the direct system parameter identification (MIMO).

To understand how these methods work, a brief explanation of the complex exponential and Ibrahim time domain methods is given next.

The Complex Exponential Method

This method takes the IRF, and assumes that it can be described by a series of complex exponentials:

$$h_{jk}(t) = \sum_{r=1}^{2N} {}_rA_{jk} e^{s_r t} \qquad [1]$$

where ${}_rA_{jk}$ is the residue of mode r, related to the complex modal constant $s_r = -\omega_r \xi_r + i\omega_r \sqrt{(1-\xi_r^2)}$ and ω_r and ξ_r are the natural frequency and viscous damping ratio of mode r, respectively.

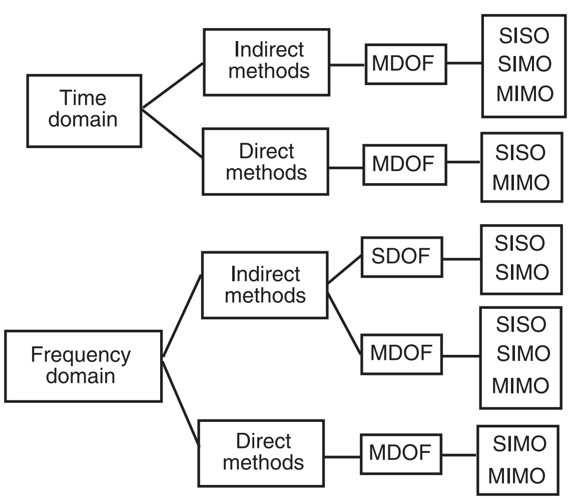

Figure 1 Classification of parameter extraction methods.

Eqn [1] is the inverse Fourier transform of the corresponding FRF:

$$H_{jk}(\omega) = \sum_{r=1}^{2N} \frac{{}_rA_{jk}}{\omega_r \xi_r + i\left(\omega - \omega_r \sqrt{(1-\xi_r^2)}\right)} \qquad [2]$$

From eqn [1], by taking L equally-spaced time intervals and simplifying the notation, one has:

$$\begin{aligned} h_0 &= h(0) = \sum_{r=1}^{2N} A_r \\ h_1 &= h(\Delta t) = \sum_{r=1}^{2N} A_r V_r \\ &\vdots \\ h_L &= h(L\Delta t) = \sum_{r=1}^{2N} A_r V_r^L \end{aligned} \qquad [3]$$

where A_r is an abbreviation for ${}_rA_{jk}$ and where $V_r = \exp(s_r \Delta t)$. As the roots, s_r, for an underdamped system occur in complex conjugate pairs, so also do the variables, V_r, and thus there must exist a polynomial with real coefficients, β, such that:

$$\beta_0 + \beta_1 V_r + \beta_2 V_r^2 + \cdots + \beta_L V_r^L = 0 \qquad [4]$$

It is possible to show that, for each V_r, $\sum_{j=0}^{L} \beta_i h_i = 0$. From this, taking the appropriate number of data points, the values of β can be obtained. Then, V_r are calculated from eqn [4] and so the natural frequencies and damping ratios are determined. From eqn [3], the values of the residues are easy to calculate, and from these, so are the modal constants. The above method is not iterative, and so no initial estimates for the modal parameters are necessary. The uncertainty lies in determining the correct number of participating modes. To determine that quantity, an iterative procedure is used, varying the number of assumed modes. When the error between the real measurements and the synthesized ones has a minimum, the correct number of modes is determined.

The Ibrahim Time Domain Method

This method, an example of a SIMO approach, uses free responses instead of IRFs. For a system with N DOFs, the free response of the structure at a point i and for the instant of time t_j can be expressed as a summation of the individual responses of each mode:

$$x_i(t_j) = \sum_{r=1}^{2N} \psi_{ir} e^{s_r t_j} \qquad [5]$$

where ψ_{ir} is the ith component of the complex eigenvector $\boldsymbol{\psi}_r$. Considering q response locations and L time instants leads to an expression of the type:

$$\underset{(q\times L)}{\mathbf{X}} = \underset{(q\times 2N)}{\boldsymbol{\Psi}} \underset{(2N\times L)}{\boldsymbol{\Lambda}} \qquad [6]$$

where $\boldsymbol{\Lambda}$ is composed of the various $e^{s_r t_j}$ elements. Considering a second set of L data points, shifted one interval Δt with respect to the first, it is possible to write a similar expression:

$$\underset{(q\times L)}{\hat{\mathbf{X}}} = \underset{(q\times 2N)}{\hat{\boldsymbol{\Psi}}} \underset{(2N\times L)}{\boldsymbol{\Lambda}} \qquad [7]$$

Defining $\mathbf{A}_S$ of order q such that $\mathbf{A_S}\boldsymbol{\Psi} = \hat{\boldsymbol{\Psi}}$ leads to $\mathbf{A_S X} = \hat{\mathbf{X}}$, from which one can calculate $\mathbf{A_S}$. in a least-squares sense. As each vector $\hat{\boldsymbol{\psi}}_r = \boldsymbol{\psi}_r e^{s_r \Delta t}$, a standard eigenproblem is obtained:

$$\left[\mathbf{A_S} - e^{s_r \Delta t}\mathbf{I}\right]\boldsymbol{\psi}_r = \mathbf{0} \qquad [8]$$

From the eigenvalues and eigenvectors, the modal parameters are evaluted. The difficulty here is to determine the order q of the problem, as the system will have as many resonances as the number of chosen responses, q, and this is quite an arbitrary choice. If q is higher than the true number of resonances, some of the solutions of eqn [8] will not be physically meaningful, they will just be computational solutions which are a priori difficult to distinguish from the genuine ones. The quality of the results can be checked using the modal confidence factor, based on repetition of the calculations in different time interval shifts: the expected value of $\hat{\boldsymbol{\psi}}_r$ from $\hat{\boldsymbol{\psi}}_r = \boldsymbol{\psi}_r e^{s_r \Delta t}$ is then compared with the calculated value of $\hat{\boldsymbol{\psi}}_r$ in the following time interval.

Frequency-domain Methods

There are many different frequency-domain parameter extraction methods. Here, the following will be addressed: the circle-fitting method (SDOF, SISO), the rational fraction polynomial (MDOF, SISO), and the identification of structural system parameters method (MDOF, SIMO), which is a direct method.

The Circle-Fitting Method

This is one of the simplest and most popular methods in the frequency domain category. The receptance FRF of an N degree-of-freedom system with hysteretic damping is given by the following expression:

$$H_{jk}(\omega) = \sum_{r=1}^{N} \frac{{}_rC_{jk}}{\omega_r^2 - \omega^2 + i\eta_r\omega_r^2} \qquad [9]$$

where η_r and ${}_rC_{jk} = \left(C_r e^{i\phi_r}\right)_{jk}$ are the hysteretic damping loss factor and the complex modal constant associated with mode r, respectively. Expression [9] is the equivalent of eqn [2], which referred to viscous damping. In this method, each mode is treated separately and the contribution of the neighboring modes to the particular one under study is assumed to be a constant, eqn [9] is therefore approximated by:

$$H_{jk}(\omega) \approx \frac{{}_rC_{jk}}{\omega_r^2 - \omega^2 + i\eta_r\omega_r^2} + {}_rD_{jk} \qquad [10]$$

where ${}_rD_{jk}$ is a complex constant associated with mode r. It can be shown that the Nyquist plot (real part versus imaginary part) of $1/\left(\omega_r^2 - \omega^2 + i\eta_r\omega_r^2\right)$ is a circle. From eqn [10], it is clear that the multiplication by the complex constant ${}_rC_{jk}$ means a magnification or reduction of the circle radius, as well as a rotation. ${}_rD_{jk}$ corresponds to a simple translation of the whole circle.

When representing the measured data in a Nyquist plot, the complete curve will not be an exact circle around each natural frequency, but eqn [10] tells us that the curve will approach a circle around those frequencies, as illustrated in **Figure 2**.

The modal parameters associated with mode r are derived from fitting a circle to the frequency response

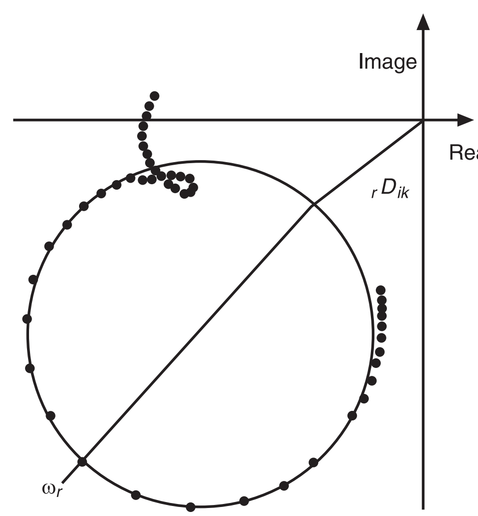

Figure 2 Nyquist plot of the receptance, showing the SDOF circle-fit approach.

curve near each natural frequency, ω_r. This objective is achieved through the use of a least-squares technique. Once the circle centre coordinates and radius are known, the natural frequency can be estimated, as it occurs where the angular frequency spacing has its highest value. The damping factor is evaluated from a generalized version of the half-power points formula. From the diameter of the circle and its position, the complex modal constant is calculated. As the process continues to the next resonance peak, the effect of the previously-identified modes is subtracted, in order to enhance the mode one is dealing with at each time.

The Rational Fraction Polynomial Method

In this method, the FRF is expressed in terms of a ratio of two polynomials (in place of the summation of simple fractions) as:

$$H(\omega) = \frac{\sum_{k=0}^{2N-1} a_k (i\omega)^k}{\sum_{k=0}^{2N} b_k (i\omega)^k} \qquad [11]$$

Defining a linearized error function between the measured FRF values and the model, and minimizing it, a linear system of equations is obtained, from which the coefficients a_k and b_k are evaluated. As such a system is usually ill-conditioned, the problem is reformulated in terms of orthogonal polynomials. Knowing the resulting coefficients, the modal parameters are retrieved. A global version of this method also exists, i.e. its SIMO version, where several FRFs are taken into account simultaneously.

The Identification of Structural System Parameters Method

This is a direct method, i.e., it provides the system matrices (**K**, **M** and **C**) which constitute the spatial model directly, from the response model (the FRFs), without computing the modal parameters. The general idea, common to all direct methods, is to take the dynamic equilibrium equation and write it for various frequency data points, in order to form an overdetermined system of equations, from which the system matrices are computed. Such a system will look like:

$$\begin{aligned} \left[-\omega_1^2\mathbf{M} + i\omega_1\mathbf{C} + \mathbf{K}\right]\overline{\mathbf{Y}}_1 &= \mathbf{F} \\ \left[-\omega_2^2\mathbf{M} + i\omega_2\mathbf{C} + \mathbf{K}\right]\overline{\mathbf{Y}}_2 &= \mathbf{F} \\ \vdots \qquad\quad & \vdots \\ \left[-\omega_L^2\mathbf{M} + i\omega_L\mathbf{C} + \mathbf{K}\right]\overline{\mathbf{Y}}_L &= \mathbf{F} \end{aligned} \qquad [12]$$

where $\bar{\mathbf{Y}}_i$ is the vector of complex response amplitudes, at frequency ω_i, which is the frequency data point, varying from 1 to L.

The Unified Matrix Polynomial Approach

The unified matrix polynomial approach (UMPA) is a method that shows that a considerable number of methods in both time and frequency domains are just particular cases of a more general polynomial formulation. That work helps us to understand better the similarities amongst most of the existing techniques, giving a more general panorama on the subject, and it contributes to a deeper knowledge of the philosophy of parameter extraction methods.

Methods to Check the Quality of Extracted Parameters

Almost every method has its own check for the quality of the obtained parameters. As mentioned before, in the complex exponential method, the correct number of modes is determined when the error between the measured and synthesized FRFs has a steep drop. In the Ibrahim time domain method, as already explained, the modal confidence factor is used, repeating the calculations in different time intervals. In the rational fraction polynomial method it is also usual to make the calculations taking each time a different set of data points and checking for the repeatability and variation of the results. In the identification of structural system parameters method, the singular value decomposition technique is usually employed to determine the effective number of degrees-of-freedom.

Another very popular method nowadays is the so-called stabilization chart. Some tolerance values are assigned *a priori* to each modal parameter (as a percentage) and the calculations are repeated taking each time a different number of modes. The chart then shows the progress of the results, indicating (by the display of a symbol) close to each peak which modal parameters have stabilized along the various iterations. Other methods for quality checking are the so-called mode indicator functions (MIFs) that indicate not only the number of existing modes, but also their relative incidence, by displaying the physical magnitude of each one and natural frequency. They can also detect repeated modes and are based on calculation of the singular value decomposition of the FRF matrix at each spectral line.

See also: **Modal analysis, experimental**, Applications; **Modal analysis, experimental**, Basic principles; **Modal analysis, experimental**, Construction of models from tests; **Modal analysis, experimental**, Measurement techniques.

Further Reading

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Construction of models from tests

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Introduction

Any real structure possesses an infinite number of degrees-of-freedom. In very simple cases, like a single beam of homogeneous isotropic material and constant cross-section, it is possible to derive an analytical solution enabling the calculation of all natural frequencies and mode shapes. In more general cases, such a solution cannot be found and therefore the modeling must be based either on a numerical study or on experimental tests undertaken on the real structure. In either case, it is necessary to decide on the number of degrees-of-freedom one must choose, so that the final model represents the reality as accurately as possible, i.e., the dynamic behavior of the structure under some loading conditions is well predicted when compared with its real behavior in service.

The decision on the appropriate number of degrees-of-freedom is called discretization, as we go from a continuous situation to a discrete one. Such a decision has to be balanced, taking into account the objectives of the study and also the available calculation power. For instance, to evaluate the dynamic behavior of a vehicle in order to assess the passengers' comfort, it may be irrelevant to discretize the whole structure in thousands of degrees-of-freedom. A few dozen may be enough.

Building a theoretical model of a structure based on a numerical approach, usually through a finite element discretization, is not a big problem nowadays, especially if the structure is a lightly damped one: the resulting stiffness and mass matrices are – most of the time – adequate to represent its real behavior with sufficient accuracy, at least over a limited frequency range. However, if the frequency range is considerably large, it may be important to consider more complicated theories or elements with higher complexity. In any case, a numerical model is never perfect and in general it is necessary to validate (see **Model updating and validating**) it with experimental tests and, in most situations, to correct it or update it using those tests as a reference. There are specific methods, known as updating techniques, to achieve the most correct model. The important issue here is to have accurate experimental results. The following presents some issues related to the construction of models from experimental results, and how these models relate to theoretically-derived ones.

Construction of a model from experimental results

Response Models vs Spatial Models

Let us consider a structure whose behavior can be well represented, for our application, by a four-degrees-of-freedom model as shown in **Figure 1**. It is therefore assumed that the dynamic behavior of the

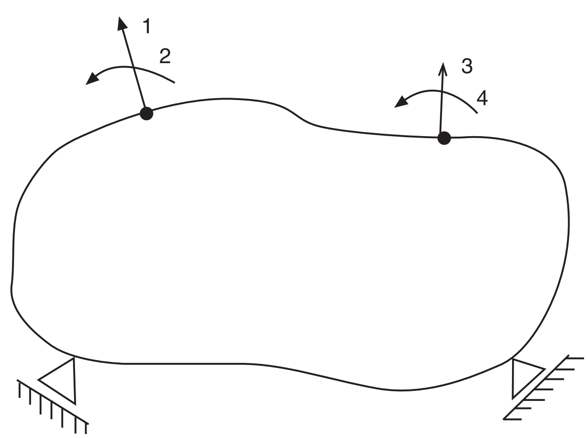

Figure 1 Real structure characterized by coordinates 1–4.

system is described with enough accuracy for our purposes using only those four-degrees-of-freedom. It is also known from the study of the single-degree-of-freedom system that the function which most conveniently reflects the behavior of that system is the particular version of the transfer function, known as a frequency response function (FRF), relating the obtained harmonic output to a given harmonic input. If F is the input, X the output and H the FRF, $X = HF$. For a system discretized by N degrees-of-freedom, the FRF properties can be contained in an $N \times N$ FRF matrix relating the amplitudes of the input and output:

$$\underset{(N\times1)}{\mathbf{X}} = \underset{N\times N}{\mathbf{H}}\,\underset{(N\times1)}{\mathbf{F}} \qquad [1]$$

For the example of **Figure 1**, eqn [1] turns into:

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \qquad [2]$$

where X_2 and X_4 are, in fact, rotational responses (angles) and F_2 and F_4 are moments.

It is worth noting that the FRF matrix is generally symmetric, due to Maxwell's reciprocity theorem, i.e., the response at degree-of-freedom (DOF) i due to an input at DOF j is equal to the response at DOF j when the same input happens at coordinate i.

The response model is available once the FRFs in $\mathbf{H}$ have been measured.

Equivalent to eqn [1] is the following inverse relationship:

$$\mathbf{F} = \mathbf{H}^{-1}\mathbf{X} \qquad [3]$$

or:

$$\mathbf{F} = \mathbf{Z}\mathbf{X} \qquad [4]$$

where $\mathbf{Z}$ is called the dynamic stiffness matrix, which for an undamped system (the damped case will be discussed later) is:

$$\mathbf{Z} = \mathbf{K} - \omega^2\mathbf{M} \qquad [5]$$

where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices, respectively, and ω is the frequency of the applied forces. If the model is represented in terms of $\mathbf{K}$ and $\mathbf{M}$ (and of the damping matrix if this is to be considered), it is called the spatial model.

Expanding [4] in accordance to the given example, it follows that:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix} \qquad [6]$$

Apparently, it would be easier to measure $\mathbf{Z}$ directly, from which one could evaluate $\mathbf{K}$ and $\mathbf{M}$ to obtain the spatial model directly. However, it is $\mathbf{H}$ that is measured in practice and then $\mathbf{Z}$ must be derived from: $\mathbf{Z} = \mathbf{H}^{-1}$. Let us see why this is the correct procedure to calculate $\mathbf{Z}$. First of all, it is clear that $Z_{ij} \neq H_{ij}^{-1}$; retrieving [6], the first equation is given by:

$$F_1 = Z_{11}X_1 + Z_{12}X_2 + Z_{13}X_3 + Z_{14}X_4 \qquad [7]$$

Consider, for example, the interpretation of Z_{12}. The individual dynamic stiffness coefficient, Z_{12}, is the force F_1 when $X_2 = 1$ and $X_1 = X_3 = X_4 = 0$. In theory, this is obvious, but in practice it simply cannot be achieved, as it implies that X_1, X_3 and X_4 should be blocked, something impossible to achieve in an experimental set up. In contrast, from [2], the first equation is:

$$X_1 = H_{11}F_1 + H_{12}F_2 + H_{13}F_3 + H_{14}F_4 \qquad [8]$$

It is very easy to calculate H_{12}, for example, as H_{12} equals X_1 when $F_2 = 1$ and $F_1 = F_3 = F_4 = 0$, or $H_{12} = X_1/F_2|_{F_1,\,F_3,\,F_4=0}$. Therefore, to calculate matrix $\mathbf{H}$ it is sufficient to apply a force at one point at a time, and to measure all the responses. After knowing $\mathbf{H}$, it is possible to calculate $\mathbf{Z}$ from its inverse ($\mathbf{H}^{-1}$). Note that as $\mathbf{Z}$ and $\mathbf{H}$ are both functions of frequency, and so the inverse has to be taken at each frequency point along the frequency range of interest, i.e., there will be as many inversions as frequency data points.

At this point, there is a conclusion: in practice, the system is characterized by the response model.

The Modal Model and its Relationship to the Spatial and Response Models

The undamped equilibrium equation for free vibration of an N-degree-of-freedom system leads to a generalized eigenvalue and eigenvector problem:

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\boldsymbol{\phi} = \mathbf{0} \qquad [9]$$

This provides a set of N natural frequencies, ω_r, and N mass-normalized mode shapes, $\boldsymbol{\phi}_r$. From the orthogonality properties for mass-normalized mode shapes, it is known that:

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathbf{I} \qquad [10a]$$

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \boldsymbol{\omega}_r^2 \qquad [10b]$$

where $\mathbf{\Phi}$ is the modal matrix whose columns are the mass-normalized mode shapes and $\boldsymbol{\omega}_r^2$ is a diagonal matrix composed of the squares of the natural frequencies. When the model is characterized in terms of $\boldsymbol{\omega}_r^2$ and $\mathbf{\Phi}$, it is called the modal model. Therefore, from knowledge of $\mathbf{K}$ and $\mathbf{M}$ (the spatial model) it is possible to obtain the modal model. The opposite is also possible, as from [10a] and [10b]:

$$\mathbf{M} = \mathbf{\Phi}^{-\mathrm{T}}\mathbf{\Phi}^{-1} \qquad [11a]$$

$$\mathbf{K} = \mathbf{\Phi}^{-\mathrm{T}}\omega_r^2\mathbf{\Phi}^{-1} \qquad [11b]$$

Therefore, one can derive the modal model from the spatial model and vice-versa.

To calculate $\mathbf{Z}$ from $\mathbf{H}$ along the frequency range of interest may imply a considerable computational effort, due to the inversions referred to at the end of the previous section. An alternative approach consists of expressing $\mathbf{Z}$ in terms of the modal parameters – natural frequencies and mode shapes.

As $\mathbf{Z} = \mathbf{K} - \omega^2\mathbf{M}$, upon substitution of [11a] and [11b], it follows that:

$$\mathbf{Z} = \mathbf{\Phi}^{-\mathrm{T}}\boldsymbol{\omega}_r^2\mathbf{\Phi}^{-1} - \omega^2\mathbf{\Phi}^{-\mathrm{T}}\mathbf{\Phi}^{-1} \qquad [12]$$

or:

$$\mathbf{Z} = \mathbf{\Phi}^{-\mathrm{T}}\left(\boldsymbol{\omega}_r^2 - \boldsymbol{\omega}^{-2}\right)\mathbf{\Phi}^{-1} \qquad [13]$$

Clearly, from [13]:

$$\mathbf{H} = \mathbf{Z}^{-1} = \mathbf{\Phi}\left(\boldsymbol{\omega}_r^2 - \boldsymbol{\omega}^{-2}\right)^{-1}\mathbf{\Phi}^{\mathrm{T}} \qquad [14]$$

and one has a direct relationship between the modal and response models. Note that eqn [13] only requires the inverse of the modal matrix, which is trivial as it is diagonal. Thus, along the frequency range it is no longer necessary to invert a matrix at each frequency point.

It is now clear how the three models are interrelated. However, it is not so clear as to which model to use, when or why. **Figure 2** illustrates this point for an N-degree-of-freedom system. Note that there are two main routes. On the one hand, there is the theoretical route, where the starting point is the analytical or numerical solution, establishing the spatial model; then, through an eigensolution, the natural frequencies and mode shapes are obtained, constituting the modal model; finally, the response model may be evaluated. On the other hand, there is the experimental route, where the departure point is the response model, formed by the relevant measured FRFs; through an identification process, which is an inverse problem, the modal model is built; using the orthogonality relationships, the spatial model may be recovered.

The necessity of permutation among the three models is to bring together the theoretical and experimental results for comparison, validation or updating objectives.Although quite illustrative, **Figure 2** may look too optimistic, as one may think that everything is perfect, all models are (more or less) easily interrelated and one can readily move from one to the other in any sense. In practice, problems may arise. In what follows, some of those problems are mentioned.

Damped Systems and Complex Modes

Any real structure is dissipative. The dissipation of energy is modeled through a damping term that in most cases in structural dynamics is either the hysteretic damping model (also called structural damping: forces proportional to the displacements but in phase with velocity) or the viscous damping model (forces proportional to the velocities). If the structure is very lightly damped, then the undamped model may be a good approximation. If that is not the case, then a damping matrix must be defined, together with the stiffness and mass matrices. The equilibrium equation becomes, for the viscous damping case, as follows:

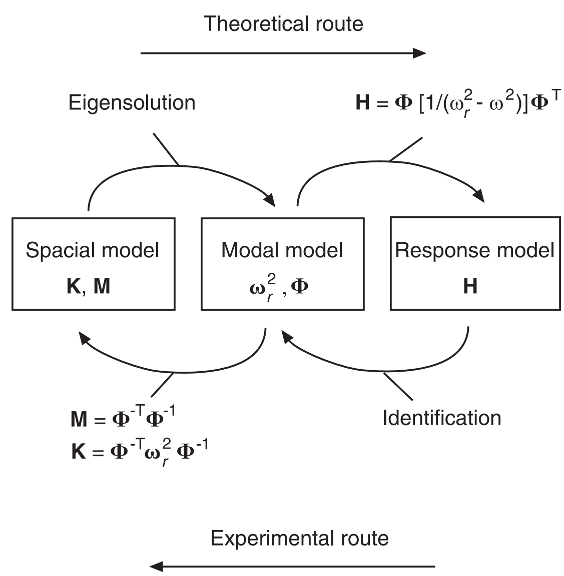

Figure 2 Interrelation among dynamic models.

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \qquad [15]$$

and for the hysteretic damping case, as:

$$\mathbf{M}\ddot{\mathbf{x}} + i\mathbf{D}\mathbf{x} + \mathbf{K}\mathbf{x} = \mathbf{f} \qquad [16]$$

Matrices $\mathbf{M}, \mathbf{K}$ and $\mathbf{C}$ (or $\mathbf{D}$) now constitute the spatial model in the damped case.

Except for very special cases (for instance, if the viscous damping matrix can be defined as a linear combination of the mass and stiffness matrices), the result of the associated eigenproblem is a set of complex natural frequencies and complex mode shapes, which define in that case the modal model. Although not so easy as the undamped case, passing from the spatial to the modal model and to the response model (the theoretical route) is not a problem. The big problem in that case is how to define the damping matrix. That is why in a pure analytical or numerical approach we only have a description of the system in terms of $\mathbf{K}$ and $\mathbf{M}$, i.e., the numerical solution is the undamped one.

To model the damping, one needs to follow the experimental route where, through adequate identification methods, we are able to go from the measured response model to the modal model, with information about the natural frequencies, damping factors and complex mode shapes. For instance, in the hysteretic damping case, each FRF can be shown to relate to the modal model through the following expression:

$$H_{kj} = \sum_{r=1}^{N} \frac{\phi_{rk}\phi_{rj}}{\omega_r^2 - \omega^2 + i\eta_r\omega_r^2} \qquad [17]$$

where H_{kj} is the measured FRF relating the response at coordinate k with the input force at coordinate j, N is the number of degrees-of-freedom, ϕ_{rk} and ϕ_{rj} the mode shape elements k and j of mode r (complex in general), ω_r the natural frequency and η_r the damping ratio of mode r.

The problem is that in many cases one wishes to validate the theoretical model or even to update it and on the one hand there is a set of real modes coming from the numerical solution, on the other hand a set of complex modes. This gives rise to an old problem: how to relate complex modes to real (or normal) modes? How to pass from one to the other? How to measure the complexity of the modes? The problem of having in practice considerably complex modes may be difficult to address. In some cases, linear transformations between real and complex modes are made, where the real modes are used as a basis to expand complex modes. These approximations, however, should be used with great care, as they are bound to produce erroneous or misleading results. Fortunately, in many applications, the complexity of the modes is not so high and even not genuine, i.e., it can result from other problems that have not been carefully addressed, such as the existence of aliasing, leakage, measurement noise, nonlinearity, identification errors, etc. Careful measurement procedures can significantly reduce the degree of false complexity indicated in measured mode shapes.

Turning to the issue of going from the measured response model to the modal model, we saw that eqn [17] is the kind of expression we often use as a basis for the identification of the modal properties of a system. While the damping ratios and natural frequencies are global properties of the system, i.e., they have the same values no matter which FRF H_{kj} is considered, the mode-shapes have a local nature, as each point has its own amplitude (and phase) (this is why most of the identification techniques proceed in two steps: first, the calculation of the global properties and second, the evaluation of the local ones). In the limit, we would appear to need only one FRF to obtain ω_r and η_r but all the FRFs to estimate ϕ_{rk} and ϕ_{rj}. In fact, neither of these two extremes are used. For the natural frequency and damping ratio we usually take more than just a single FRF, as variations always occur and experience advises to take a set of FRFs to obtain a kind of 'average result'. For the mode-shapes, not all the FRFs are necessary, because there are some inter-relationships that hold true and reduce the number of responses that need to be measured. If one measures a point FRF (excitation and measurement at the same DOF), say H_{kk}, the numerator of [17] becomes $(\phi_{rk})^2$ and ϕ_{rk} is therefore evaluated. Measuring a transfer FRF, H_{kj}, allows for the calculation of the product $\phi_{rk}\phi_{rk}$, known as a modal constant of mode r. As ϕ_{rk} is already known, we obtain ϕ_{rj}. As a consequence, it is not necessary anymore to measure H_{jj}, as its numerator (ϕ_{rk}^2) can now be computed. The implication of such properties of the mode-shape components, known as consistency properties, is that we only need to measure one column (or one row) of the FRF matrix $\mathbf{H}$ to obtain the whole mode-shape (or modal) matrix.

It is often recommended to measure more than a single column, to obtain some redundant data to improve our confidence in the results, especially when we suspect the existence of repeated natural frequencies. Theoretically, based on the consistency properties, we could even obtain the whole FRF matrix. For instance, for a 3×3, FRF matrix, let us suppose we measured H_{11}, H_{21} and H_{31}. Due to the symmetry of the matrix, we already know H_{12} and

H_{13}. So, we miss H_{22}, H_{23} ($=H_{32}$) and H_{33}. Due to the consistency properties, with H_{11} and H_{12} we could recover H_{22}; with H_{11} and H_{13}, we would obtain H_{33}; with H_{22} and H_{33}, we would obtain H_{23}.

Unfortunately, the evaluation of unmeasured FRFs based on a single column (or row) of the FRF matrix is not so straightforward as it seems, due to the incompleteness of the model in terms of frequency range, as we will see next.

Complete and Incomplete Models

To ensure clarity of exposition, the undamped case is used once more. The issue of moving to and from the different spatial, modal and response models is not the real problem, even when taking the experimental route, provided that appropriate methods are used. Probably the greatest problem of all is the incompleteness of the models obtained from the experimental tests. From the numerical solution, one has a model with N-degrees-of-freedom, which in some applications can be of the order of 10^4 or even more. Assuming that this N is representative of the behavior of the structure, the model with matrices of order N will be considered as the complete one. In contrast, from the experimental point of view, it is not usually possible to measure all the coordinate motion responses, or to apply all force excitations. Normally, only a few dozen degrees-of-freedom are measured. The system is therefore a reduced or incomplete one by comparison with the theoretical version.

Another 'source' of incompleteness is related to the measured frequency range. Only a few modes are covered and are therefore 'identifiable' from tests, in contrast with the theoretical model. To identify in an accurate way the modes within the frequency range of interest, one has to take account of the influence of the modes outside that range, the so-called low- and high-frequency residuals.

This sort of incompleteness precludes the exact evaluation of the complete FRF matrix from measurements of a single column, as explained before, since the residual mode-shapes do not verify the consistency properties. So, as in practice this is always the case, we cannot effectively reconstruct unmeasured FRFs, but can only use such properties to estimate unmeasured mode-shapes within the frequency range of interest.

If the incompleteness has only to do with the lack of measured coordinates, the relation between the modal and response models given by eqn [14] becomes:

$$\underset{(n\times n)}{\mathbf{H}} = \underset{(n\times N)}{\boldsymbol{\Phi}} \left(\underset{N\times N}{\boldsymbol{\omega}_r^2 - \boldsymbol{\omega}^2}\right)^{-1} \underset{(N\times n)}{\boldsymbol{\Phi}^{\mathrm{T}}} \qquad [18]$$

where $n < N$ is the number of measured coordinates. If the incompleteness is also in the number of modes available, say $m < N$, eqn [18] becomes:

$$\underset{(n\times n)}{\mathbf{H}} = \underset{(n\times m)}{\boldsymbol{\Phi}} \left(\underset{(m\times m)}{\boldsymbol{\omega}_r^2 - \boldsymbol{\omega}^2}\right)^{-1} \underset{(m\times n)}{\boldsymbol{\Phi}^{\mathrm{T}}} \qquad [19]$$

Note that for $\mathbf{H}$ to be of full rank the number of retained modes must be equal or larger than the number of coordinates, i.e., $m \geq n$. Inverting [18] or [19] produces a reduced stiffness matrix $\mathbf{Z}^{\mathrm{R}}$.

In the case where $n = m$, expressions [11a] and [11b] can be used to go from the modal to the spatial model, where all matrices are $n\times n$. $\mathbf{M}$ and $\mathbf{K}$ are in that case reduced mass and stiffness matrices. As already mentioned, it is often necessary to bring together the experimental and theoretical models, for validation or updating purposes, for example. As the order of magnitude of both models is very different, the comparison between the two is only possible either by a condensation of the theoretical model or by expansion of the experimental one. In either case, there are several techniques available.

For the condensation of $\mathbf{K}$ or $\mathbf{M}$, the most used method is that known as Guyan reduction: suppose one wishes to represent an N-degree-of-freedom system only through p (primary) coordinates, corresponding to the dimension of the experiments. The remaining s $(= N - p)$ coordinates are called secondary (or slave) and in those it is supposed that there are no applied forces. In a static case, where $\mathbf{KX} = \mathbf{F}$, $\mathbf{K}$ is partitioned so that:

$$\begin{bmatrix} \mathbf{K}_{pp} & \mathbf{K}_{ps} \\ \mathbf{K}_{sp} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_p \\ \mathbf{x}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_p \\ \mathbf{0} \end{Bmatrix} \qquad [20]$$

The objective is to obtain a condensed (or reduced) $\mathbf{K}^{\mathrm{R}}$ matrix referred to the p coordinates, i.e.,

$$\mathbf{K}^R \mathbf{x}_p = \mathbf{f}_p \qquad [21]$$

Eliminating $\mathbf{x}_s$ in [20], it follows that:

$$\mathbf{K}^R = \mathbf{K}_{pp} - \mathbf{K}_{ps}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sp} \qquad [22]$$

A similar formula can be found for the reduced mass matrix.

For the expansion process, the most used technique is that known as Kidder's expansion, although several interpolation alternatives are also possible. Kidder's expansion is somehow the reverse of the Guyan reduction, although normally used together with the mass matrix. From the partitioned free vibration equilibrium equation, it follows that:

$$\left[\left[\begin{array}{c|c}\mathbf{K}_{pp} & \mathbf{K}_{ps}\\ \hline \mathbf{K}_{sp} & \mathbf{K}_{ss}\end{array}\right]-\omega_r^2\left[\begin{array}{c|c}\mathbf{M}_{pp} & \mathbf{M}_{ps}\\ \hline \mathbf{M}_{sp} & \mathbf{M}_{ss}\end{array}\right]\right]\left\{\begin{array}{c}\boldsymbol{\phi}_p\\ \hline \boldsymbol{\phi}_s\end{array}\right\}_r=\left\{\begin{array}{c}\mathbf{0}\\ \hline \mathbf{0}\end{array}\right\} \quad [23]$$

where $(\boldsymbol{\phi}_p)_r$ are the known (measured) coordinate amplitudes and $(\boldsymbol{\phi}_s)_r$ the unmeasured ones, for each mode, r. It is possible to show that:

$$(\boldsymbol{\phi}_s)_r = -\left[\mathbf{K}_{ss} - \omega_r^2\mathbf{M}_{ss}\right]^{-1}\left[\mathbf{K}_{sp} - \omega_r^2\mathbf{M}_{sp}\right](\boldsymbol{\phi}_p)_r \quad [24]$$

Whenever trying to compare both theoretical and experimental models, several authors advise the use of expansion instead of condensation, so that most of the information in the theoretical model is preserved.

See also: **Modal analysis, experimental**, Applications; **Modal analysis, experimental**, Basic principles; **Modal analysis, experimental**, Measurement techniques; **Modal analysis, experimental**, Parameter extraction methods; **Model updating and validating**.

Further Reading

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Applications

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Introduction

This final article in the series on experimental modal analysis is concerned with the applications to which the mathematical models derived from the test data are to be put. These applications fall into two categories: those which use the resulting models in a qualitative, primarily visual, way and those which use the models in a quantitative, numerical way. The first set of applications is primarily concerned with relatively practical, troubleshooting, problem-solving activities and, historically, constitutes the original purpose of most early modal tests. More recently, as the quality and reliability of the derived models have improved, several more ambitious applications have emerged in which the models are used for numerical predictions of various alternative configurations or scenarios to the test set-up. These include:

- comparisons of theoretically- and experimentally-derived models with a view to improving the analytical model (model 'validation' or 'updating');
- structural modification and optimization activities;
- structural assembly analysis;
- operating conditions response prediction and force determination; and
- measurement of dynamic properties of complex materials and structures, including for damage detection purposes.

The following sections provide an introduction to these applications, discussing the main requirements and limitations of each.

It should be noted at the outset that different applications make use of the derived mathematical model in different forms: spatial, modal or response. The troubleshooting applications generally use the derived models in their original format – as modal models. The model validation and updating applications use both modal and spatial models: modal models for the initial comparisons and spatial models at the stage when modeling errors are sought, and corrections to the original model are introduced. Structural modification and assembly applications generally rely more on response models, as do the applications concerned with operating response and excitation analysis.

Troubleshooting

In troubleshooting applications, the engineer is generally interested in obtaining visual displays of the individual modes of vibration, or even of the operating deflection shapes ('modes', but not 'normal modes'). By inspection of these mode shapes, it is often possible to develop an appreciation and physical understanding of exactly how, and why, the structure is vibrating. A skilled engineer can often diagnose troublesome vibration conditions from such a visual inspection of the structure's modes and, eventually, prescribe appropriate modifications or other corrective actions. However, all this inspection and interpretation is done in an essentially qualitative way, and relatively few computations (if, indeed, any at all) are involved.

Theoretical Model Validation

Probably the single most practiced application of modal testing is that of validating a theoretical model using measured data. 'Validation' means checking that an already-available theoretical model (usually, a finite element model) is in fact 'valid'. By this we mean that the model is capable of predicting the actual dynamic behavior of the test structured to an acceptable accuracy. The last qualification is important because it is not realistic to expect the theoretical model to be absolutely correct.

The various phases of model validation can be listed as follows:

- Comparison
- Correlation
- Verification
- Reconciliation
- Error location or 'localization'
- Model updating or 'correction'

and it will be helpful to explain each of these in a few words:

- 'Comparison' is the passive process of setting measured and predicted results side by side and observing their similarities and differences.
- 'Correlation' is the ensuing step of quantifying these differences in order to assess the extent of the comparison. A number of standard numerical indicators are widely used in this process.
- 'Verification' is a procedure which should be carried out before any attempt is made to explain and correct for the differences which have been observed between the test-derived model and its analytical counterpart. Verification is the process of determining whether a given model is capable of describing the behavior of the subject structure, if all the individual model parameters are assigned the correct values. A model may not be verified if it lacks certain features or freedoms which are present in the actual structure since, in this case, no amount of parameter correction can compensate for the errors embedded in the basic model.
- 'Reconciliation' refers to the sometime delicate task of establishing reasons for the observed differences between measurement and prediction. These differences can arise either because of inaccurate models, or because of imperfect measurements, and the necessary corrective action is, of course, different in the two cases. A third explanation is also frequently-encountered: the actual structure under test might not conform precisely to the specification (or drawings) upon which the theoretical model will have been based.
- 'Error location' is the very difficult task of establishing which elements or parts of the model contain the errors (whether these be errors in the theoretical model or in the test data). This is made difficult because of the usually heavily underdetermined nature of the analysis: there will be many more parameters with potential errors in the model than there are independent measured data from the modal test with which to locate them.
- 'Updating' is the final step in the complete validation process: it is the computation of the necessary changes to be introduced to the parameters which have been identified as containing the errors.

Comparison and Correlation of Vibration Properties

The most common form of analysis-test comparison is of the modal properties. Plots of measured vs predicted natural frequencies, such as that shown in **Figure 1A** and side-by-side comparisons of mode shapes, shown in **Figure 1B**, are fairly standard displays. It is also possible to compare measured with predicted frequency response functions (FRFs), but these are generally more difficult to interpret (**Figure 1C**). An alternative way of displaying the mode shape comparison in **Figure 1B** is to plot the values of the respective eigenvectors (the deflected mode shape display is simply one way of plotting the elements in a mode shape vector), test vs analysis, such as the example shown in **Figure 1D**.

The plot of natural frequencies in **Figure 1A** can be used to obtain a quantitative measure of the agreement between test and analysis, principally by determining the slope of the straight line drawn through the plotted points. This should be unity and any deviation in this slope, accompanied by minimal scatter of the points, suggests strongly that there is simply an error in some global property, such as modulus or density, but that the basic model is in close agreement with the measurements. If, on the other hand, the points are markedly scattered about the best-fit line, then that suggests a poor correlation between test and prediction. However, it is necessary to exercise some caution in this type of comparison: a point should only be plotted on the graph in **Figure 1A** after it has been established that the shape of the mode corresponding to the measured natural frequency matches closely that of the mode which

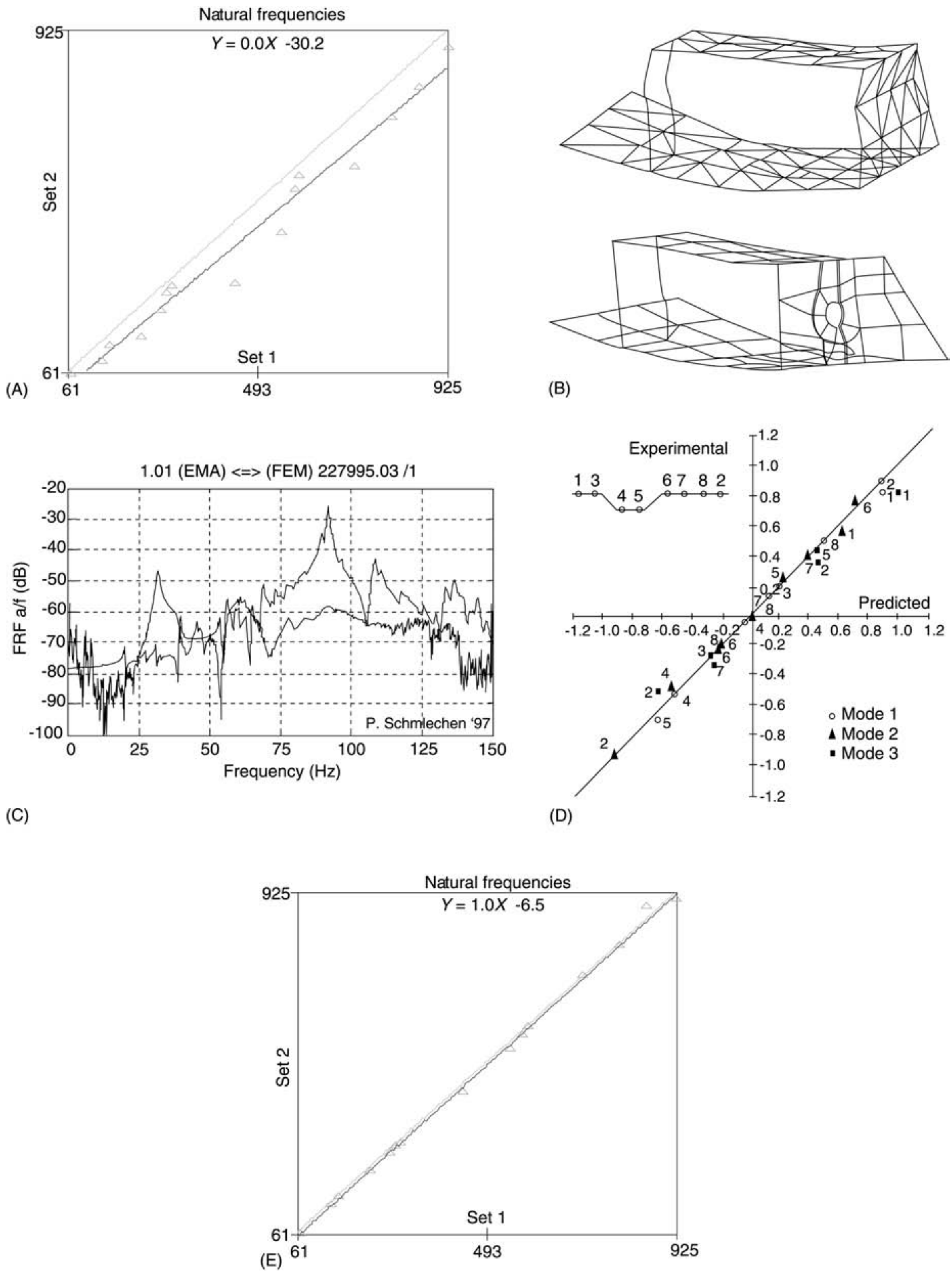

Figure 1 (A) Plot of measured vs predicted natural frequencies. (B) Measured and predicted mode shapes. (C) Measured and predicted FRFs. (D) Comparison of mode shape vectors. (E) A more reliable plot of measured vs predicted natural frequencies.

corresponds to the relevant predicted natural frequency. It is not valid to plot the natural frequency of, say, a torsion mode against a predicted natural frequency of a bending mode – just because they happen to have similar frequency values. In fact, it is essential that correlated mode pairs are identified prior to constructing a lot such as that shown in **Figure 1A** and this can only be achieved by a systematic analysis of the mode shapes of the experimental and theoretical modes.

The necessary systematic comparison between two mode shapes (such as measured vs predicted) can be made based on the plot shown in **Figure 1D** which shows the corresponding elements from the two vectors plotted one against the other. In the ideal case, all such points should lie on a straight line of slope ± 1. If they lie close to a straight line of a different slope, this indicates very similar mode shapes, but different scaling, in the two cases. From such a plot two average parameters can be extracted: the slope of the best-fit straight line drawn through the points (known as the 'modal scale factor', MSF) and the scatter of those points about that line (known as the 'mode shape correlation coefficient' or 'modal assurance criterion', MAC). The relevant formulae for these two quantities are:

$$\mathrm{MSF}(X, A) = \frac{\sum_{j=1}^{n} (\psi_X)_j (\psi_A)_j^*}{\sum_{j=1}^{n} (\psi_A)_j (\psi_A)_j^*}$$

and

$$\mathrm{MAC}(A, X) = \frac{\left|\boldsymbol{\psi}_X^{\mathrm{T}} \boldsymbol{\psi}_A\right|^2}{\left(\boldsymbol{\psi}_X^{\mathrm{T}} \boldsymbol{\psi}_X\right)\left(\boldsymbol{\psi}_A^{\mathrm{T}} \boldsymbol{\psi}_A\right)}$$

respectively. If two sets of eigenvectors are compared, say measured modes vs predicted modes, then a matrix of MAC coefficients can be produced, and can be displayed in several ways, including a table of numerical values (**Figure 2A**) or a diagram (**Figure 2B**).

From the diagram in **Figure 2B**, it can be seen that the correlated modes (those with a MAC exceeding 80%) do not correspond exactly with a direct sequential comparison. Thus the natural frequency of test mode 1 (SET 1) should not be compared numerically with the natural frequency of analysis mode 1 (SET 2) because they do not relate to the same basic mode, as determined by the shapes. The first valid comparison that can be made is between test mode 1 (SET 1) and analysis mode 2 (SET 2). Then, 2;3, 3;4, 5;5, 6;6, ... Once this pairing has been completed, the natural frequency plot can be redrawn this time making a more reliable comparison between test and analysis, see **Figure 1E**. Clearly, the earlier plot in **Figure 1A** of uncorrelated modes is misrepresentative of the true degree of correlation between test and analysis.

The MAC correlation function is widely used as the first level of quantitative comparison between two sets of modal data. The diagrams and tables shown in **Figure 2** can, however, become confused as a result of incompleteness of the measured mode shape data. To check for such problems, it is possible to compute an AUTOMAC, in which a set of eigenvectors are correlated with themselves. The result of such a computation should be a simple diagonal matrix, indicating that each mode shape vector correlates only with itself. If anything different is found, then it must be concluded that the modes are described by an insufficient number of degrees of freedom (DOFs) or points – a situation which can occur all too easily in experimental situations where the number of measured points is usually severely restricted for reasons of resource. Examples of both situations are shown in **Figures 2C** and **2D**, which show the same data plotted using 102 measured DOFs in **Figure 2C**, but only 30 DOFs in **Figure 2D**.

The most recent format for plotting all these data is shown in **Figure 2E**, in which the natural frequency plots of **Figure 1**, and the mode shape correlation in the MAC and AUTOMAC diagrams in **Figure 2**, are combined into a single plot that contains all the relevant information referred to as the FMAC.

Model Updating and Error Localization

The Basic Principle of Model Updating

Correlation constitutes the essential prerequisite for model updating, or correction. In model updating, the concern here is with the logical conclusion of comparison or correlation: namely, the discovery and correction of errors in the theoretical model (although it is also possible that the reasons for the two sets of data to differ might not always lie in the theoretical model: it is possible that certain of the measured data might contain errors). However, whether the actual errors reside in the theoretical model or in the test data, the task is essentially the same: to find the regions of the model/structure where the causes of the less-than-perfect correlation are located. Once these regions have been identified, then detailed examination of the specific data usually reveals which model contains the errors. If they are found to be in the measured data, then a partial re-test is the ideal solution, although if this is not feasible,

Analytical mode number	Experimental mode number 1	2	3	4	5	6	7	8	9	10
1	100	0	1	0	0	0	0	0	0	0
2	0	100	1	1	0	0	0	0	0	0
3	0	1	94	3	2	0	0	0	0	0
4	0	0	2	92	5	3	0	0	0	0
5	0	0	0	4	86	7	4	0	0	0
6	0	0	0	0	7	81	9	5	0	0
7	0	0	0	0	0	10	75	10	5	0
8	0	0	0	0	0	0	12	71	11	5
9	0	0	0	0	0	0	0	14	68	11
10	0	0	0	0	0	0	0	0	16	65

(A) *Modal assurance criterion (MAC) %*

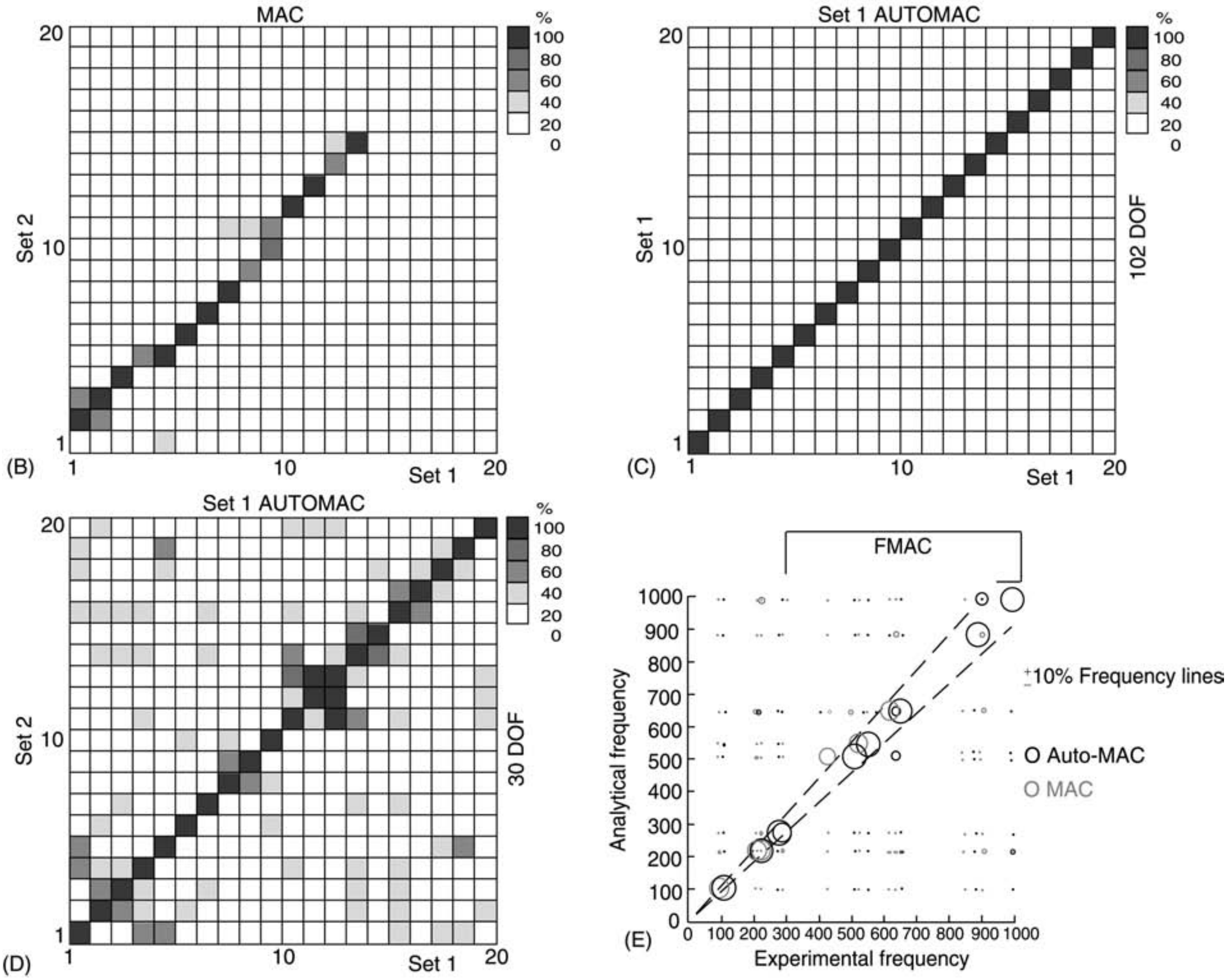

Figure 2 (See Plate 46). (A) Correlation of test and analysis. (A) MAC table; (B) MAC diagram; (C) AutoMAC – many DOFs; (D) AutoMAC – reduced DOFs; (E) FMAC plot.

then the offending data need to be eliminated from the whole validation process. If, on the other hand, the errors are believed to be contained in the theoretical model, then it is necessary to determine what adjustments to the numerical values of the critical model parameters are required in order to bring the theoretical model predictions closer into line with the measured data.

The algorithms and procedures to carry out the tasks summarized in the above paragraph are rather complex, and cannot be covered in an introductory article such as this. However, there is a complete entry devoted to the topic of model updating and the interested reader is directed there for a more complete description (see **Model updating and validating**). This section simply discusses some of the more practical considerations.

The Processes Involved in Model Updating

Essentially, what is required in the process of updating (or 'refining', or 'correcting') a theoretical model is to identify changes that need to be made to the individual elements in the mass and stiffness coefficients in the spatial model. This is often translated into the requirement to update or correct elements in the mass and stiffness matrices which constitute the numerical representation of the spatial model.

Verification The updating task comprises one introductory and two active stages. Before embarking on an updating exercise, it is important to determine the extent to which the two models (experimental and analytical) are different as there is little point in seeking to correct for discrepancies that lie within the measurement uncertainty that must be applied to the test data. At the same time, it is also necessary to establish that the analytical model is capable of representing the measured behavior, even when numerical adjustment of its various parameters has been undertaken. In fact, satisfying this condition is by no means a foregone conclusion: if the theoretical model is too simple, and contains only a few DOFs while the actual test structure is complex and has many active DOFs, then it is clear that no amount of adjusting such a theoretical model can render it capable of replicating the behavior of a much more complicated structure. In this respect, the theoretical model which is to be updated must first be 'verified' as being intrinsically capable of describing the structure's dynamic behavior. Only then is it appropriate to search for the parameters in the model which need to be adjusted, and then to introduce the appropriate numerical changes to those parameters.

Error location Once this precondition is satisfied, the first of the two main updating tasks is to locate those regions or elements in the model which need to be adjusted. This is, in fact, the most difficult task and it is made so by the fact that the measured data that are to be used to perform this task are not only inaccurate (as are all measured data) but – and more importantly – they are generally heavily incomplete. In most modal tests on real structures, practical considerations mean that only a relatively small number of modes of vibration are measured, and these are defined at only a small number of DOFs, as compared with the total number of DOFs in the theoretical model. As a result, the error location problem is heavily underdetermined, which means that it is almost impossible to obtain a unique solution – many 'solutions' are generally found – and those which are obtained may well be ill-conditioned because of the inadequacy of the measured data.

Model adjustment Nevertheless, there are several algorithms that seek to identify the most critical of the parameters in the model and, once identified, the second task is relatively straightforward – generally using some optimization criteria, a 'best' set of individual correction factors for the identified erroneous elements can be determined. This process, also, suffers from the underdetermination feature refered to above and care must usually be exercised not to allow model parameters to be modified to a degree which cannot be justified from a physical point of view. This is very important because, in many cases, a satisfactory match between measured data and predictions from an updated model can be achieved at the expense of realism in the adjusted theoretical model. Such a situation constitutes a numerical solution, but not a reasonable one from a physical or practical viewpoint. Any updated model must be able to pass a test of 'reasonableness' in its numerical parameters.

Algorithms for Model Updating

There have been many algorithms proposed for the task of updating a theoretical model, but only two or three survive. The most popular one is an iterative one which is based on the sensitivity properties of the subject model; sensitivity to small changes in individual mass or stiffness elements in that model. **Figure 3** illustrates a typical updating session spanning some 10 iterations, showing: (a) the current degree of correlation (MAC matrix); (b) the evolution of the selected parameters being updated; (c) the evolution of the MAC values for the correlated mode pairs and (d) the discrepancies between measured and predicted natural frequencies (one of the primary indicators of agreement between two models). However, not all updating sessions are as classical as the one illustrated here.

Structural Modification and Structural Assembly Analysis

The next important application for a modal test-derived model of a structure is a family of methods which are referred to as 'modification' and 'assembly' methods. Although these two families of applications appear to be quite distinct, in fact they share the same

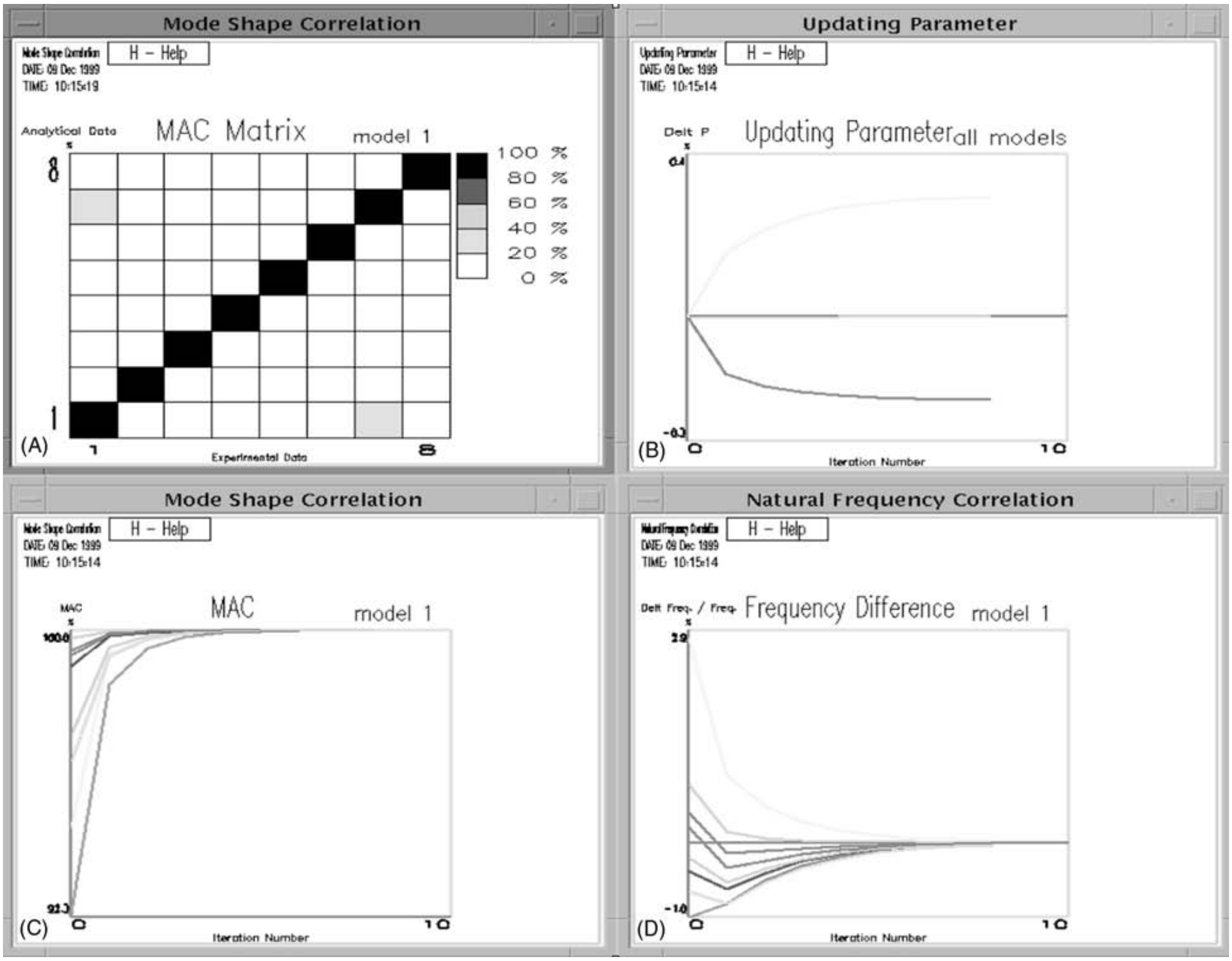

Figure 3 (See Plate 49). Example of evolution of model updating session. (A) Current MAC matrix; (B) evolution of updated parameters; (C) evolution of MAC values for correlated model pairs; (D) evolution of natural frequency discrepancies.

analytical techniques and processes. Indeed, it is quite reasonable to visualize a 'modification' as a simple case of one substructure being connected to the main test structure to form an 'assembled structure'.

The Three Specific Applications

There are three well-defined versions of this class of problem. The first – simple structural modification – seeks to answer the following question: 'if we have obtained a modal model of a particular test structure using modal testing techniques, can we readily predict (without the need for further tests) how those dynamic properties would change if a specified modification was introduced to the original test structure?'

The second application is an inverse version of the first, and can be summarized by the question: 'given a mathematical model derived from measurements on the test structure, how can we ascertain what modifications need to be made to that structure in order to change its vibration properties to some prescribed values?'

The third of these applications is more ambitious. If there are a number of components which will be combined, or coupled, to form a structural assembly, how can the dynamics of that assembly be forecast from a prior knowledge of the dynamics of each component structure individually? In other words, 'if we have a mathematical model of each separate component, (how) can we predict the dynamics of the assembly formed by coupling those components together, even before that assembly is realized physically?'

This last demand actually subsumes the first and second, and it can be seen that the generic application is structural assembly or structural coupling, and this is described pictorially in **Figure 4** which shows one component (A, with N_A DOFs) connected in a particular way to another component (B, with N_B DOFs)

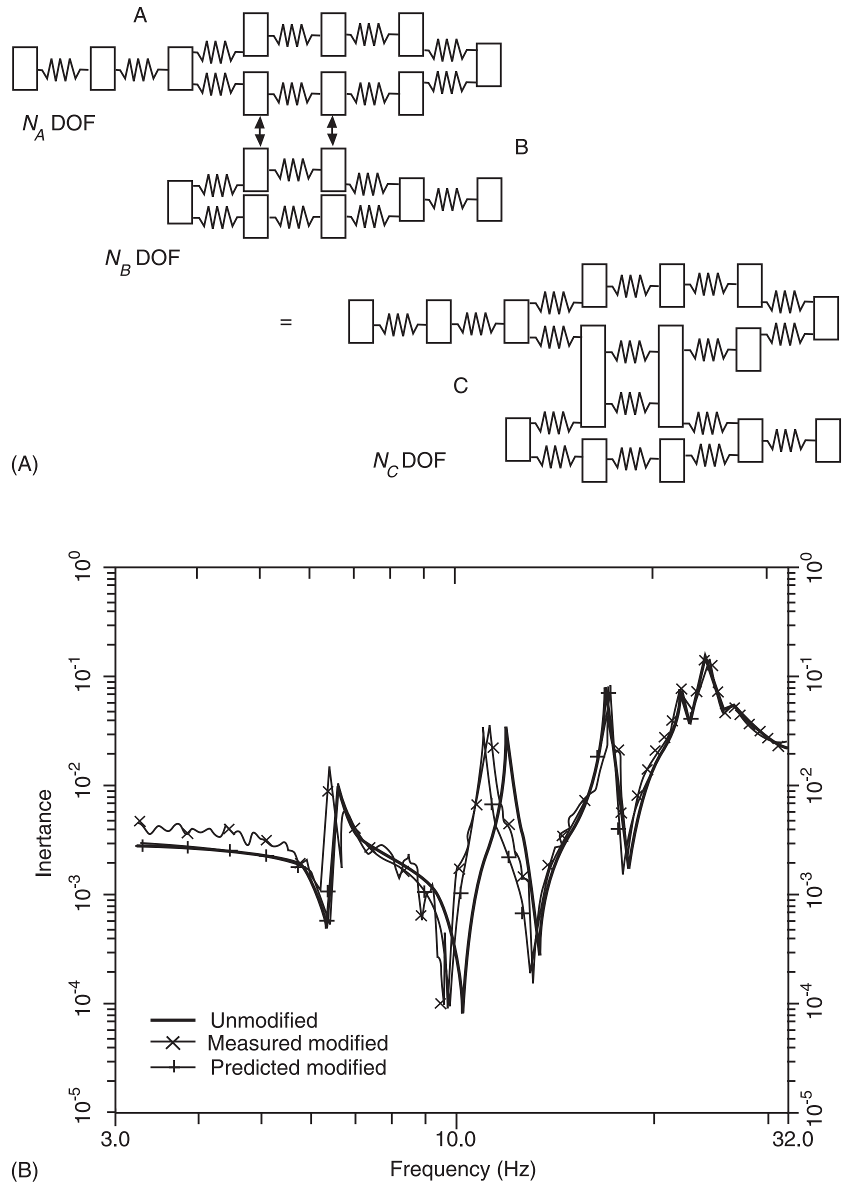

Figure 4 (A) Schematic illustration of a structural assembly formed by combining components A and B to form C; (B) practical application of structural modification.

so as to form a coupled structure, *C* (or structural assembly), with N_C DOFs. (It should be noted that N_C is not simply $N_A + N_B$.)

The methodology of structural assembly analysis seeks to provide a reliable prediction of the dynamic properties of the complete assembly (*C*) given the corresponding properties of each component individually. Thus, if we have a modal model for each of the components, *A* and *B* as separate structures, perhaps from a modal test on each, then we seek to predict the modal model for the assembly, *C*. Alternatively, if a response model exists for each of the component structures individually, then it may be preferred to derive a response model for the assembly. Both approaches are possible and are routinely made. Once again, a full discussion is not practical in this section, but there is a separate article on structural modification to which the reader who seeks more detail is directed (see **Structural dynamic modulations**).

It will suffice here to discuss some of the implications of using such analysis tools as can be found in that companion article in a practical situation. While the essential formulas and algorithms to undertake the required predictions are relatively simply stated, their successful implementation in practice is often difficult. This fact can be traced to the same issues as mentioned in the previous paragraphs: partly to the imprecision but mostly to the incompleteness of the test data used to construct (limited, incomplete) models for the individual components. In this case, the incompleteness comes in two respects. First, there is the inevitable restriction to the frequency range over which tests are conducted, and modal properties are extracted. This range is usually curtailed by the structure itself because, at higher frequencies, the modes tend to become closer and closer together with the result that they eventually become indistinguishable in the response functions that are measured and analyzed to extract the required modal properties. The second restriction arises because not all the DOFs which are active in the connection interface can generally be measured, and so some are omitted from the component models, frequently with severe consequences on the resulting application. Most common amongst this category are the rotation DOFs which are notoriously difficult to measure, yet are clearly important participants in the coupled structure interactions.

As with the earlier example of model updating, care must be exercised here when seeking to exploit these advanced and potentially powerful applications in practice. Nevertheless, when such care is judiciously applied, very useful results can be obtained and considerable further testing can be avoided.

Response Prediction and Force Determination

Next, we describe two applications which are once again linked by virtue of the common underlying mathematics used in each case, even though the physical context can be quite different in the two cases.

Operating Response Levels

One obvious application of a modal-test derived model of a structure is to use that model to predict how the test structure would respond if subjected to any or many of a range of different excitations ('different' to that used for the modal test), usually referred to as 'operating' excitations, and thus to 'operating' responses. Interest might well be focussed on using the model to simulate excitations which are either multiple, and thus difficult to apply directly to the actual test piece, or at high forcing levels, thereby carrying a risk of damaging the structure. In both cases, the test-derived model can be used to make such predictions with a high degree of reliability and at a fraction of the cost that would be incurred to obtain the same results by direct testing.

The basic theory of the method can be illustrated by the specific case of single-harmonic (but multi-point) excitation, for which the equation that yields the response, $\mathbf{X}$, in terms of a specified excitation, $\mathbf{F}$, is:

$$\mathbf{X}_{n_1\times 1}\mathrm{e}^{\mathrm{i}\omega t} = \mathbf{H}_{n_2\times n_1}(\omega)\mathbf{F}_{n_1\times 1}\mathrm{e}^{\mathrm{i}\omega t}$$

The required elements in the FRF matrix, $\mathbf{H}(\omega)$, can be derived from the modal model by the familiar formula:

$$\mathbf{H}_{n_2\times n_1}(\omega) = \mathbf{\Phi}_{n_2\times m}\left(\lambda_\mathrm{r}^2 - \omega^2\right)^{-1}_{m\times m}\mathbf{\Phi}^\mathrm{T}_{m\times n_1}$$

Thus it can be seen how knowledge of the structure's modal properties can be used to predict that structure's response to an arbitrary set of harmonic excitation forces, $\mathbf{F}$, such as are experienced under service or operating conditions. Indeed, the type of response vector that has been derived here is often referred to as an operating deflection shape, or ODS. Clearly, by invoking one of several versions of Fourier analysis, the illustrated example of harmonic excitation and response can be extended to almost any type of vibration: periodic, random, transient, etc.

Determination of Unknown Excitation Forces from Measured Responses

The preceding technique of operating response analysis, based on the modal test-derived model, is generally very reliable. Care must be taken, as always, to ensure that the model used is complete enough for the task, and that means that a sufficient number of modes, m, is used so that the elements in the FRF matrix are accurate. However, the computation is generally well-conditioned.

There is an obvious extension of the foregoing approach in the form of the inverse formulation which sets out to determine the excitation force vector, which is supposed to be unknown in this application, using knowledge of the operating response levels, which can be more readily measured in a service environment than can the excitation forces.

The theoretical principle of the approach seems straightforward enough. It is the inverse of the above formula for response prediction and can be written as:

$$\mathbf{F}_{n_1\times 1}\mathrm{e}^{\mathrm{i}\omega t} = \mathbf{H}^{-1}_{n_1\times n_2}(\omega)\mathbf{X}_{n_{21}\times 1}\mathrm{e}^{\mathrm{i}\omega t}$$

However, there is an obvious problem in using this equation because of the need to invert the rectangular FRF matrix, $\mathbf{H}(\omega)$. In fact, it is necessary to use the generalized inverse to proceed with this expression, and we can write:

$$\mathbf{F}_{n_1\times 1} = \mathbf{H}^{+}_{n_1\times n_2}(\omega)\mathbf{X}_{n_{21}\times 1}$$