Mechanics of Composite Materials and Laminates

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Chapter 1

FIBERS, MATRICES, AND COMPOSITE MATERIALS

Advanced composite materials usually consist of two constituents, i.e., the reinforcing element and the supporting matrix. The reinforcing element being much stiffer and stronger than the matrix carries the load, and the matrix provides protection and lateral support to the reinforcing element. Since materials are much stronger in fibrous form than in bulk form, many reinforcing elements in advanced composites are high performance fibers. It is conceivable that properties of fiber-reinforced composites can be tailored by the selection of fiber and matrix systems and by varying fiber volume fraction. Therefore, it is crucial to have a good understanding of available reinforcing fibers and matrices and their properties.

1.1 FIBERS

The principal reinforcement fibers used in advanced composites include boron, carbon (graphite), glass, ceramic, and aramid (Kevlar) fibers. They are used in the continuous form or chopped into various lengths. Boron, carbon, and ceramic fibers offer excellent stiffness and strength properties, while glass and aramid fibers offer high strengths and moderate stiffnesses. Table 1.1 presents typical properties of these fibers. The properties of some steel and aluminum alloys are also listed for comparison.

Boron Fibers are manufactured by depositing boron on a tungsten filament. Currently, boron fibers are produced by chemical vapor deposition from a gaseous mixture of hydrogen and boron trichloride (BCl_3) on an electrically heated (to 1250°C) tungsten substrate of 0.5 mil. (12.5 μm) diameter. The tungsten substrate is continuously pulled through the reactor. By varying the speed, the desired boron coating thickness can be achieved. Currently boron fibers are produced in sizes of 4 mils. (100 μm), 5.6 mils. (140 μm), and 8 mils. (200 μm) diameter. Boron fibers have been modified by coating them with silicon carbide (SiC) and boron carbide (B_4C) to improve the mechanical properties of their metal matrix composites.

Boron fiber exhibits very high strength and modulus properties. Unfortunately, at this time, its cost is still very high due mainly to the high cost of tungsten filament and boron trichloride. In order to lower the cost of boron fibers, the tungsten filament is replaced by a carbon monofilament. In addition, the drawing speed of the boron carbon deposition reactor is substantially increased, leading to a higher production rate and, thus, to savings.

Carbon (Graphite) Fibers are manufactured from a variety of precursors or starting material fibers. The precursor is carbonized through the use of high temperature up to 1700°C. Carbon fiber can be graphitized by heat treatment at higher temperatures for improved performance. The so called graphite fibers do not contain only graphite. In fact, they consist mainly of carbon. The amount of graphite in the fibers depends on the heat treatment temperature. The higher temperature in the final processing increases the percentage of graphite.

The popular precursors used for making graphite fibers include polyacrylonitrile (PAN), pitch, and rayon. Graphite fibers from PAN precursors have been the most popular in recent years. Most of the graphite fibers that are commercially available range in diameter from 0.3 to 0.5 mil. (7 to 13 μm). In general, cross sections vary from circular to oval shape depending on the precursor and process used.

The process by which PAN is converted into carbon fibers involves the following steps. First, the precursor is stretched to align the molecular orientation parallel to the fiber and then locked in this position by heat treatment at 220°C in air. The subsequent carbonization is carried out in an inert atmosphere at temperatures ranging from 1000°C to 1500°C during which the fibers are transformed into carbon fibers. It is during this stage that high mechanical properties are developed. The final stage involves the graphitization heat treatment at temperatures exceeding 1800°C which yields better tensile modulus by increasing the carbon content and improving the crystallite structure and preferred orientation of the graphite like crystallites.

Pitch and other similar materials are the by-products of distillation of coal and petroleum refining. The fibers are, in general, produced by first transforming the pitch into a liquid crystalline (mesophase) state for an extended period of time. This is done at 400-500°C in an inert atmosphere. The mesophase pitch is then spun into filament form and subsequently carbonized and graphitized at higher temperatures. The cost of pitch is low. However, this process is more costly than that of the PAN based graphite fibers.

Rayon is a cellulosic material obtained from wood pulp. The fibers have been used in textiles for clothing and tire cord. The process of converting rayon fiber precursors into carbon fibers requires the following steps: a low temperature stabilization heat treatment, a carbonization treatment at 1300°C, and a stretch graphitization treatment at 2800-3000°C. The tension applied during graphitization is needed to align the disordered array of graphite in the direction of the fiber. High mechanical property levels cannot be obtained without stretching. Rayon based graphite fibers are very expensive due to the extremely high temperatures required for their stretch graphitization.

Glass Fibers are still the most popular reinforcement materials for making composites due to their low cost and high strength. The two most used glass fibers are E glass (alumninoborosilicate) and S glass (magnesium aluminosilicate). S glass offers higher tensile strength and better properties at elevated temperatures. The diameters of the glass fibers range from 0.1 to 0.8 mil. (3-20 μm).

Aramid Fibers are the generic name for fibers formed from polymers. Du Pont's Kevlar fibers are most important in composites applications. Kevlar fibers are light and possess very high strength and rather high modulus. They are inherently resistant to flame and high temperature; they do not melt but will decompose at approximately 500°C. Kevlar fibers are not brittle, and yarns of these fibers can be woven into fabrics. They have the unique property in a negative coefficient of thermal expansion in the longitudinal direction.

Kevlar fibers have a relatively high moisture absorption capability (up to 5-6 percent). They are subject to degradation by ultraviolet light with the consequence of a substantial reduction in strength. However, this problem is minimized by the fact that the matrix material in the composite may shield the fibers from ultraviolet light. In general, Kevlars have good resistance to lubricants, oils, and solvents, except for some strong acids.

A few other reinforcing fibers worth mentioning are silicon carbide (SiC) fiber, silicon carbide whisker, and aluminum oxide (Al_2O_3) fiber. These fibers possess common quality in their inherent high temperature stability, thus are most suitable for metal matrix and ceramic composites which require high temperature fabrication processes.

Table 1.1 Mechanical Properties of Fibers, Matrix Materials and Conventional Metals

Material	Tensile Modulus E GPa (msi)	Tensile Strength σ_u GPa(ksi)	Density ρ g/cm^3
Fiber			
E-Glass	77.0(11)	3.5(500)	2.54
S-Glass	85.0(12)	4.5(650)	2.48
Silicon Carbide (Nicalon)	190.0(27)	2.8(400)	2.55
Carbon (Hercules AS4)	240.0(35)	3.6(510)	1.80
Carbon (Hercules IM7)	280.0(40)	5.2(750)	1.80
Carbon (Toray T300)	240.0(35)	3.5(500)	1.80
Boron	385.0(55)	3.5(500)	2.65
Aramid (Kevlar-49)	130.0(18)	3.8(550)	1.45
Aramid (Kevlar-29)	65.0(9.5)	3.8(550)	1.45
Polymeric Resin			
Epoxy	3.5-5.0(0.5-0.7)	0.05-0.13(7-20)	1.20
Polyester	2-4.5(0.3-0.07)	0.04-0.07(6-10)	1.10-1.40
Thermoplastic (PEEK)	5.0(0.7)		
Metal			
Steel	210.0(30)	0.34-2.10(50-300)	7.80
Aluminum Alloys	70.0(10)	0.14-0.62(20-90)	2.70
Titanium (Ti-6Al-4V)	110(16)	0.92(134)	4.46

1.2 MATRIX MATERIALS

Matrix materials commonly used in forming composites can be categorized into epoxy based thermoset resins, thermoplastics, ceramics, and metals. **Epoxy** resins are compatible with all types of fibers and are used for the majority of advanced composite materials. The strength of epoxy can be as high as 60 MPa (9 ksi) and the modulus greater than 3450 MPa (0.5 msi). A wide variety of epoxy resins and curing agents which can be formulated to give a broad range of properties.

Polyester resins offer the advantages of good mechanical, chemical, and electrical properties, dimensional stability, ease of handling, and low cost. Additives are easily incorporated into polyester resin systems to provide flame retardant properties, superior surface finishes, pigmentation, weather resistance, and low shrink properties. Inorganic or inert materials are often added to result in better surface appearance, moldability, and lower cost. They are popular in making reinforced plastics for automobile parts. Epoxies offer better dimensional stability and mechanical strength to weight ratios. They are not as economical as polyesters, but their excellent properties can make them the cost/performance preference in critical applications.

Polyimide resins are thermosetting resins for high temperature applications. These materials may be used at temperatures up to 350°C . At present, processing these materials is more difficult and time consuming than processing epoxy resins.

Thermoplastic polymers can be suitable for making fibrous composites. The main advantages of thermoplastics over thermoset resins are their notably high elongation to failure, unlimited self life, and the capability of being reformed at elevated temperatures. The resulting composite can be sheared, stamped, dimpled, and hot formed using adapted metal working equipment. In addition, the short process cycle time makes thermoplastic composites extremely attractive for mass production. One of the thermoplastic matrix used in advanced composites is PEEK (poly-ether-ether-ketone).

For high temperature application, metals (e.g., aluminum and titanium alloys) and ceramics are used as the matrix for composites. Metal-matrix and ceramic-matrix composites are much more difficult and expensive to process. At his time, their potential has not been fully exploited.

1.3 COMPOSITES

Composites are formed by combining reinforcing fibers and matrix materials. The properties of a composite obviously depend on the type of fiber and matrix used as well as the fiber volume fraction. Mixing different types of fibers in forming hybrid composites is also very attractive since hybrid composites may offer better properties and potential savings.

In general, the reinforcements in a composite take the form of continuous fibers or short fibers. For the former, fibers may be unidirectional or in the form of woven fabrics. Multi-directional reinforcement preforms can also be produced using various techniques such as braiding.

There are more than a dozen basic processes being utilized to fabricate composites. Among them are injection molding, compression molding, pressure bag molding, resin transfer molding, filament winding, and continuous pultrusion. Each process has its own characteristics and limitations. The selection of the fabrication process is often dictated by the resin system and part size, shape, and production rate.

Most of the epoxy based advanced composites are fabricated from unidirectional and fabric "prepregs", which are layers of unidirectional fibers or fabric impregnated with predetermined amounts of uniformly distributed resin supported by a thin backing paper. These prepreg tapes must be kept at very low temperature (0°F) to prolong their shelf life. Autoclave vacuum bag molding under heat and pressure is the most common process used in the aerospace industry. The amount of resin in the prepregs can be controlled coupled with the controlled pressure in the curing cycle to result in the desired fiber content.

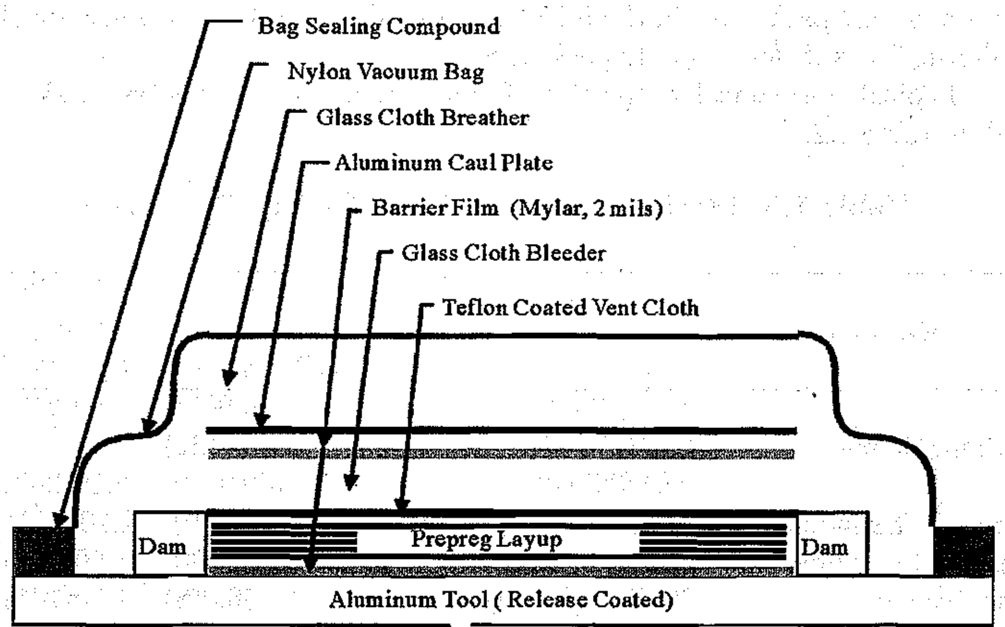


Fig. 1.1 Autoclave vacuum-bag curing

In the autoclave process, the prepreg laminates are laid up on a mold of the same configuration of the part surface to be fabricated. (See Fig. 1.1.) The laminate is cured in a nylon bag under vacuum, heat, and pressure according to a specified heating pressure cycle. After curing, the part is removed from the autoclave and allowed to cool. The part may require surface finishing to achieve the desired appearance. Coating may also be applied to provide surface toughness.

Autoclave molding is a relatively slow process and is not suitable for mass production. For mass production, the use of fast curing resins such as polyester is necessary.

Compression molding is a popular method of manufacturing composites with fast curing matrices such as SMC (sheet molding compound). The forming of sheet molding compound involves depositing chopped glass fibers (up to 5.0 cm long) on a coat of polyester resin filled paste. This process is usually automated. It is usually packaged in rolls. The starting SMC material (the charge) is placed between matched molds and subsequently subjected to heat and pressure. The curing cycles range from less than a minute to about five minutes. This molding method is particularly suitable for mass production. SMC composites are of particular interest to automobile manufacturers due to their short cycle time and good structural properties.

Resin Transfer Molding (RTM) is a promising cost saving manufacturing method for making large and complex composite parts. The preshaped dry fiber reinforcement is placed inside the tool cavity. Liquid resin is then pumped or transferred into the tool to impregnate the reinforcement, which is subsequently cured according the cure cycle recommended for the resin. Apparently, the viscosity of the resin is a crucial parameter in the RTM process. The resin flow may also displace the prealigned fibers if not properly performed.

Typical mechanical properties of some advanced composites, and SMC are listed in Table 1.2.

Table 1.2 Mechanical Properties of Fiber Composites

Material	Type	Tensile Modulus GPa (msi)	Tensile Strength GPa (ksi)	Density g/cm^3
	T300/5208	180.0(26)	1.50(210)	1.55
Carbon/Epoxy	IM7/3501-6	150.0(22)		1.55
	AS4/3501-6	140.0(20)	2.10(300)	1.55
Carbon/Thermoplastic	AS4/PEEK (APC-2)	140.0(20)	2.10(300)	1.57
Boron/Aluminum	B/A1 2024	210.0(30)	1.50(210)	2.65
Glass/Epoxy	S2 Glass/Epoxy	56.0(8)	1.70(245)	1.80
Aramid/Epoxy	Kev 49/Epoxy	70.0(10)	1.40(200)	1.40
SMC	R-50*	15.0(2.0)	0.15(21)	2.0

*$R-50$ indicates that the composite contains 50% finer by weight.

Chapter 2
ELASTICITY OF ANISOTROPIC MATERIALS

2.1 INDEX NOTATION AND TENSORIAL TRANSFORMATION

The symbol x_i with the range $i = 1, \cdots, n$ is used to denote any element in the set $\{x_1, x_2, \cdots, x_n\}$. The symbol i is called an index. Similarly, notations with multiple indices such as a_{ij} $(i,\ j = 1, ..., n)$ are used to represent individual components in the set of $n \times n$ elements $\{a_{11},\ a_{12},\ \cdots,\ a_{nn}\}$.

In using index notations, one often encounters the following equation

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = a \tag{2.1}$$

which can be interpreted as the scalar product of two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. This equation can also be written as

$$\sum_{i=1}^{3} x_i y_i = a \tag{2.2}$$

A **summation convention** is usually used to express (2.2) in the simple form

(only comes in pairs)

eg - $x_i y_i z_i$ not in convention

$$x_i y_i = a\ ,\ i = 1, 2, 3 \qquad \text{(Einstein's Sum)} \tag{2.3}$$

The *summation convention* states that the repetition of an index in a product term indicates a summation over the range of that index. The repeated index is called a **dummy index**. An index that is not summed is referred to as a **free index**. Since a dummy index does not carry additional meaning besides a summation operation, any index can be used without changing its result. For example, $x_i y_i$ and $x_j y_j$ represent the same quantity.

When there are more than two summation operations to be performed, great caution must be exercised in the use of the summation convention. For example, consider the following equation

$$x_i y_i w_j v_j = a \tag{2.4}$$

where the two dummy indices i and j must be distinguished. A term with three or more repeated indexes has no meaning unless it is used to denote a specific term with no summation.

Coordinate Transformation

Consider two Cartesian coordinates (x_1, x_2, x_3) and (x_1', x_2', x_3'). A base vector is a unit vector parallel to a coordinate axis. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the base vectors for the x_1, x_2, x_3 coordinate system, and $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ be the base vectors for the x_1', x_2', x_3' system. Since the coordinate axes are mutually orthogonal, we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{2.5}$$

and

$$\mathbf{e}_i' \cdot \mathbf{e}_j' = \delta_{ij} \tag{2.6}$$

where a dot indicates scalar product and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.7}$$

is the Kronecker delta.

A vector $\mathbf{x}$ can be projected onto the two coordinate systems with the result:

$$\mathbf{x} = x_j' \mathbf{e}_j' = x_j \mathbf{e}_j \tag{2.8}$$

Taking the scalar product of (2.8) with $\mathbf{e}_i'$, we obtain

$$x_j' \mathbf{e}_j' \cdot \mathbf{e}_i' = x_j \mathbf{e}_j \cdot \mathbf{e}_i' \tag{2.9}$$

By using the relation (2.6), (2.9) yields

$$x_j' \delta_{ij} = x_j \mathbf{e}_j \cdot \mathbf{e}_i' \tag{2.10}$$

By defining

$$\beta_{ij} = \mathbf{e}_i' \cdot \mathbf{e}_j \tag{2.11}$$

(2.10) can be written as

$$x_i' = \beta_{ij} x_j \tag{2.12}$$

A similar procedure by taking a scalar product with $\mathbf{e}_i$ leads to the inverse relation

$$x_i = \beta_{ji} x_j' \tag{2.13}$$

Equation (2.12) or (2.13) gives the coordinate transformations between the two sets of components of a vector $\mathbf{x}$ in the two coordinate systems.

Substitution of (2.13) into (2.12) yields

$$x_i' = \beta_{ij} \beta_{kj} x_k' \tag{2.14}$$

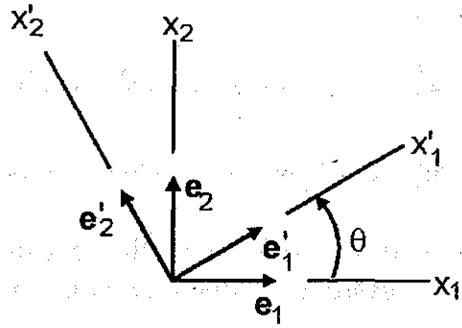


Figure 1

from which we conclude that

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix} \qquad \beta_{ij}\beta_{kj} = \delta_{ik} \quad \begin{cases} k = j \\ j = i \end{cases} \tag{2.15}$$

In a similar manner, we can show that

$$\beta_{ji}\beta_{jk} = \delta_{ik}$$

In matrix form, (2.15) can be written as

$$[\beta]\,[\beta]^T = [I] \tag{2.16}$$

where superscript T indicates the transposed matrix and $[I]$ is the 3×3 identity matrix. Thus, $[\beta_{ij}]$ is orthogonal.

Example 2.1

In the two dimensional space, the base vectors for (x_1, x_2) and (x'_1, x'_2) coordinate systems shown in Fig. 2.1 are given by

$$\mathbf{e}_1 = (1, 0)\ ,\ \mathbf{e}_2 = (0, 1)$$

and

$$\mathbf{e}'_1 = (\cos\theta, \sin\theta)\ ,\ \mathbf{e}'_2 = (-\sin\theta, \cos\theta)$$

respectively. The coordination transformation matrix can be computed from (2.11) as

$$\begin{aligned} \beta_{11} &= \mathbf{e}'_1 \cdot \mathbf{e}_1 = \cos\theta\ ,\ \beta_{12} = \mathbf{e}'_1 \cdot \mathbf{e}_2 = \sin\theta \\ \beta_{21} &= \mathbf{e}'_2 \cdot \mathbf{e}_1 = -\sin\theta\ ,\ \beta_{22} = \mathbf{e}'_2 \cdot \mathbf{e}_2 = \cos\theta \end{aligned}$$

Thus,

$$x'_1 = \beta_{1j}x_j = x_1\cos\theta + x_2\sin\theta$$

$$x'_2 = \beta_{2j}x_j = -x_1\sin\theta + x_2\cos\theta$$

The above equations are readily recognized as the coordinate transformation between the original system and the system obtained by rotating the original system counter clockwise with an angle θ.

Scalar, Vector and Tensor

Let (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) be two fixed sets of rectangular Cartesian coordinates related by the transformation law

$$x'_i = \beta_{ij}x_j$$

where β_{ij} is defined by (2.11).

A system of quantities is called a scalar, a vector, or a tensor depending on how the components of the system are defined in the variables x_1, x_2, x_3 and how they are transformed when the variables x_1, x_2, x_3 are changed to x'_1, x'_2, x'_3.

A system is called a scalar if it has only a single component Φ in the variables x_i and a single component Φ' in the variables x'_i and if Φ and Φ' are numerically equal at the corresponding points

$$\Phi(x_1, x_2, x_3) = \Phi'(x'_1, x'_2, x'_3) \tag{2.17}$$

A system is called a vector field or a tensor field of order one if it has three components ξ'_i in the variables x'_i and if the components are related by the transformation law

$$\begin{aligned} \xi'_i(x'_1, x'_2, x'_3) &= \beta_{ik}\xi_k(x_1, x_2, x_3) \\ \xi_i(x_1, x_2, x_3) &= \beta_{ki}\xi'_k(x'_1, x'_2, x'_3) \end{aligned} \tag{2.18}$$

The tensor field of order two is a system which has nine components t_{ij} in the variables x_1, x_2, x_3 and nine components t'_{ij} in the variables x'_1, x'_2, x'_3, and the components are related by the characteristic transformation law

$$\begin{aligned} t'_{ij} &= \beta_{im}\beta_{jn}t_{mn} \\ t_{ij} &= \beta_{mi}\beta_{nj}t'_{mn} \end{aligned} \tag{2.19}$$

We obtain from generalization to tensor of order n:

$$
\begin{aligned}
t'_{ijk\cdots} &= \beta_{im}\beta_{jn}\beta_{kp\cdots}t_{mnp\cdots} \\
t_{ijk\cdots} &= \beta_{mi}\beta_{nj}\beta_{pk\cdots}t'_{mnp\cdots}
\end{aligned} \tag{2.20}
$$

Note that if all components of a tensor vanish in one coordinate system, then they vanish in all other coordinate systems.

Contraction

Contraction is an operation on tensors that equates two indexes and sums over that index, e.g.,

$$A_{ijkl\cdots} \rightarrow A_{iikl\cdots} \tag{2.21}$$

Contraction of a tensor of order n will yield a tensor of rank $n-2$. For example, consider a tensor of the second order A_{ij} which follows the transformation law, i.e.,

$$A'_{ij} = \beta_{im}\beta_{jn}A_{mn} \tag{2.22}$$

Taking contraction over the two indexes, we obtain

$$
\begin{aligned}
A'_{ii} &= \beta_{im}\beta_{in}A_{mn} \\
&= \delta_{mn}A_{mn} \\
&= A_{mm}
\end{aligned}
$$

This result indicates that A_{ii} is a scalar and, thus, is invariant with respect to coordinate transformation.

Partial Derivatives

If $x'_i = \beta_{ij}x_j$, then for a vector v_j we have

$$v'_j\left(x'_1, x'_2, x'_3\right) = \beta_{jk}v_k\left(x_1, x_2, x_3\right) \tag{2.23}$$

Differentiating both sides of the equation, we obtain

$$
\begin{aligned}
\frac{\partial v'_j}{\partial x'_i} &= \beta_{jk}\frac{\partial v_k}{\partial x'_i} = \beta_{jk}\frac{\partial v_k}{\partial x_m}\frac{\partial x_m}{\partial x'_i} \\
&= \beta_{jk}\beta_{im}\frac{\partial v_k}{\partial x_m}
\end{aligned} \tag{2.24}
$$

This says that partial derivatives of any tensor field behave like the components of a Cartesian tensor. It should be noted that this is not true in curvalinear coordinate systems.

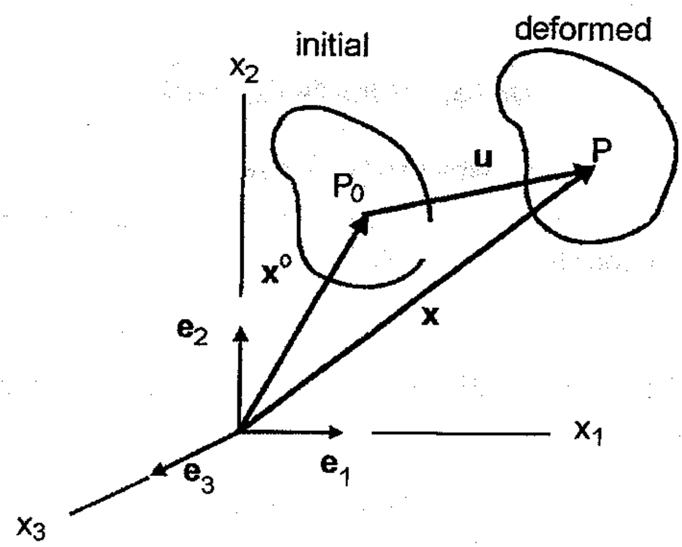


Figure 2

Quotient Rule

Consider the following equation

$$A_{ijk}B_{jk} = C_i$$

The quotient rule states that if any two quantities in the above equation are tensors, then the third quantity must also be a tensor.

2.2 STRAIN

When a solid is subjected to external loads, material points in the body are displaced. If two material points experience a change in the distance between them, then deformation is present. Displacements that do not result in changes in distance between any two material points are called rigid body motions or rotations.

Consider a body in the initial state and the deformed state (see Fig. 2.2). The positions of a material point in these two states are given by the position vectors x_i^0 and x_i, respectively. The relation between these two positions is given by

$$x_i = x_i\left(x_1^0, x_2^0, x_3^0\right) \quad i = 1, 2, 3 \tag{2.25}$$

If the body remains as an integral body, the relation given by the above equations must be unique and can be inverted to

$$x_i^0 = x_i^0\left(x_1, x_2, x_3\right) \tag{2.26}$$

The displacement that the material point travels from the initial state to the deformed state is given by the displacement vector

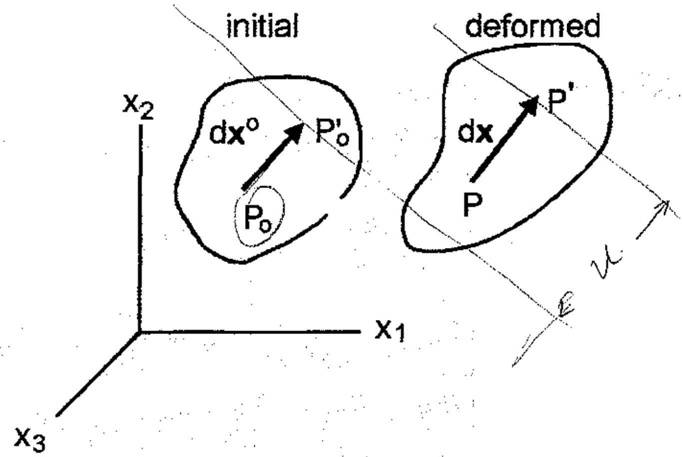


Figure 3

$$u_i = x_i - x_i^0 \tag{2.27}$$

The displacement components u_i can be expressed as functions of the **Lagrangian variables** (x_i^0) or the **Eulerian variables** (x_i).

Let P_0' be a material point in the neighborhood of P_0. The position of P_0' is $x_i^0 + dx_i^0$ where dx_i^0 represents the difference in positions of P_0 and P_0' as shown in Fig. 2.3. After deformation, the difference in positions of these two material points is denoted by dx_i. The initial and final distances between these two material points are given by

$$ds_0^2 = dx_i^0 dx_i^0 = \left(dx_1^0\right)^2 + \left(dx_2^0\right)^2 + \left(dx_3^0\right)^2 \tag{2.28}$$

and

$$ds^2 = dx_i dx_i = \left(dx_1\right)^2 + \left(dx_2\right)^2 + \left(dx_3\right)^2 \tag{2.29}$$

respectively. A measure of the change of the distance between these two neighboring points after deformation can be given by $ds^2 - ds_0^2$.

Noting that

$$dx_m = \frac{\partial x_m}{\partial x_i^0} dx_i^0 = \left(\frac{\partial x_m^0}{\partial x_i^0} + \frac{\partial u_m}{\partial x_i^0}\right) dx_i^0 = \left(\delta_{im} + \frac{\partial u_m}{\partial x_i^0}\right) dx_i^0$$

$dx_2 = \frac{\partial x_2}{\partial x_2^0} dx_i^0$

we can write

$$\begin{aligned} ds^2 - ds_0^2 &= dx_m dx_m - dx_m^0 dx_m^0 \\ &= \frac{\partial x_m}{\partial x_i^0}\frac{\partial x_m}{\partial x_j^0} dx_i^0 dx_j^0 - \delta_{ij} dx_i^0 dx_j^0 \\ ? &= \left(\frac{\partial x_m}{\partial x_i^0}\frac{\partial x_m}{\partial x_j^0} - \delta_{ij}\right) dx_i^0 dx_j^0 \\ &= \left[\left(\delta_{im} + \frac{\partial u_m}{\partial x_i^0}\right)\left(\delta_{jm} + \frac{\partial u_m}{\partial x_j^0}\right) - \delta_{ij}\right] dx_i^0 dx_j^0 \\ &= \left[\frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0} + \frac{\partial u_m}{\partial x_i^0}\frac{\partial u_m}{\partial x_j^0}\right] dx_i^0 dx_j^0 \\ &= 2E_{ij} dx_i^0 dx_j^0 \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left(\frac{\partial x_m}{\partial x_i^0}\frac{\partial x_m}{\partial x_j^0} - \delta_{ij}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0} + \frac{\partial u_m}{\partial x_i^0}\frac{\partial u_m}{\partial x_j^0}\right) \end{aligned} \tag{2.31}$$

is the **Lagrangian strain tensor.** $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

In a similar manner, we can derive

$$ds^2 - ds_0^2 = 2e_{ij} dx_i dx_j \tag{2.32}$$

in which the **Eulerian strain tensor** is defined as

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial x_m^0}{\partial x_i}\frac{\partial x_m^0}{\partial x_j}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \tag{2.33}$$

Thus, stretching (deformation) in a body after deformation can be completely described by either the Lagrangian strain tensor or the Eulerian strain tensor.

Infinitesimal Strain

In structural materials, the range of material stretching is usually limited. This condition leads to small displacement gradients, i.e.,

$$\left|\frac{\partial u_i}{\partial x_j^0}\right| << 1 \quad \text{and} \quad \left|\frac{\partial u_i}{\partial x_j}\right| << 1$$

Consequently, the product terms in the Lagrangian strain tensor and in the Eulerian strain tensor can be neglected with the result,

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0}\right) \tag{2.34}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.35}$$

These are the infinitesimal strain tensors. $\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{1}{2}\left(2\frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u}{\partial x_1}$

Noting that

$$\begin{aligned}\frac{\partial u_i}{\partial x_j^0} &= \frac{\partial u_i}{\partial x_m}\frac{\partial x_m}{\partial x_j^0} = \frac{\partial u_i}{\partial x_m}\left(\delta_{mj} + \frac{\partial u_m}{\partial x_j^0}\right) \\ &= \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_m}\frac{\partial u_m}{\partial x_j^0} \cong \frac{\partial u_i}{\partial x_j}\end{aligned} \tag{2.36}$$

we conclude that the two infinitesimal strain tensors have identical numerical values and that the distinction between the Lagrangian and Eulerian descriptions of the deformed state vanishes.

Henceforth, we will use e_{ij} to denote the infinitesimal strain components.

Physical Interpretation of Infinitesimal Strain Components

Consider the special material element represented by dx_i (the vector connecting P and P', see Fig. 2.3) which is parallel to the x_1-axis, i.e.,

$$dx_1 = ds\ ,\ dx_2 = dx_3 = 0$$

From (2.32), we have

$$ds^2 - ds_0^2 = 2e_{11}dx_1^2 = 2e_{11}ds^2 \tag{2.37}$$

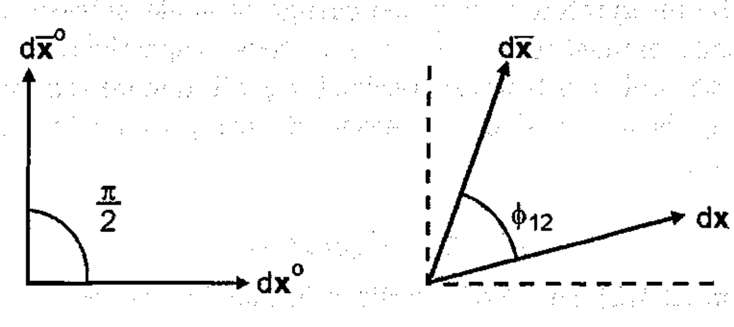


Figure 4

Since the stretching is small, we have $ds + ds_0 \cong 2ds$. From (2.37)

$$e_{11} = \frac{1}{2}\frac{ds^2 - ds_0^2}{ds^2} \cong \frac{ds - ds_0}{ds} \tag{2.38}$$

Thus, e_{11} has the same meaning as the elementary definition of uniaxial strain. Interpretations for e_{22} and e_{33} can be obtained in a similar manner.

To find the meaning of other strain components, we consider two mutually orthogonal material elements dx_i^0 and $d\overline{x}_i^0$ such that

$$dx_1^0 = ds_0 \ , \ dx_2^0 = dx_3^0 = 0$$

$$d\overline{x}_2^0 = d\overline{s}_0 \ , \ d\overline{x}_1^0 = d\overline{x}_3^0 = 0$$

After deformation, the corresponding material elements are denoted by dx_i and $d\overline{x}_i$, respectively. Denoting the angle between dx_i and $d\overline{x}_i$ by ϕ_{12} (see Fig. 2.4) and taking the scalar product of these two vectors, we obtain

$$\begin{aligned} ds d\overline{s} \cos\phi_{12} &= dx_i d\overline{x}_i = \frac{\partial x_i}{\partial x_j^0}\frac{\partial x_i}{\partial x_k^0} dx_j^0 d\overline{x}_k^0 \\ &= \frac{\partial x_i}{\partial x_1^0}\frac{\partial x_i}{\partial x_2^0} dx_1^0 d\overline{x}_2^0 \\ &= 2E_{12} ds_0 d\overline{s}_0 \end{aligned} \tag{2.39}$$

For small displacement gradients (infinitesimal strains) we have

$$E_{12} = e_{12} \ , \ ds \cong ds_0 \ , \ d\overline{s} \cong d\overline{s}_0$$

and (2.39) can be approximated by

$$e_{12} = \frac{1}{2}\cos\phi_{12} = \frac{1}{2}\sin\left(\frac{\pi}{2} - \phi_{12}\right) \cong \frac{1}{2}\left(\frac{\pi}{2} - \phi_{12}\right) \tag{2.40}$$

Thus, e_{12} can be interpreted as half the change of angle between the two material elements originally parallel to the x_1- and x_2- axis, respectively.

Henceforth, unless otherwise specified, e_{ij} will denote the infinitesimal strain tensor. Since e_{ij} is a second order tensor, its components follow the coordinate transformation law as

$$e'_{ij} = \beta_{im}\beta_{jn}e_{mn} \tag{2.41a}$$

The equations in (2.41a) can also be written in matrix form as

$$[e'] = [\beta]\,[e]\,[\beta]^T \tag{2.41b}$$

where superscript " T " indicates transpose of a matrix.

Since only six strain components are independent, the following vector notation is often used:

$$\{e\} \triangleq \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{Bmatrix} \triangleq \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{13} \\ e_{12} \end{Bmatrix} \tag{2.42}$$

The transformation relations of (2.41) can be expressed in the contracted form as

$$\{e'\} = [T_e]\{e\} \tag{2.43}$$

where

$$[T_\varepsilon] = [T_e] = \begin{bmatrix} \beta_{11}^2 & \beta_{12}^2 & \beta_{13}^2 & 2\beta_{12}\beta_{13} & 2\beta_{11}\beta_{13} & 2\beta_{11}\beta_{12} \\ \beta_{21}^2 & \beta_{22}^2 & \beta_{23}^2 & 2\beta_{22}\beta_{23} & 2\beta_{21}\beta_{23} & 2\beta_{21}\beta_{22} \\ \beta_{31}^2 & \beta_{32}^2 & \beta_{33}^2 & 2\beta_{32}\beta_{33} & 2\beta_{31}\beta_{33} & 2\beta_{31}\beta_{32} \\ \beta_{21}\beta_{31} & \beta_{22}\beta_{32} & \beta_{23}\beta_{33} & (\beta_{22}\beta_{33}+\beta_{23}\beta_{32}) & (\beta_{23}\beta_{31}+\beta_{21}\beta_{33}) & (\beta_{31}\beta_{22}+\beta_{21}\beta_{32}) \\ \beta_{11}\beta_{31} & \beta_{12}\beta_{32} & \beta_{13}\beta_{33} & (\beta_{13}\beta_{32}+\beta_{12}\beta_{33}) & (\beta_{11}\beta_{33}+\beta_{13}\beta_{31}) & (\beta_{11}\beta_{32}+\beta_{12}\beta_{31}) \\ \beta_{11}\beta_{21} & \beta_{12}\beta_{22} & \beta_{13}\beta_{23} & (\beta_{12}\beta_{23}+\beta_{13}\beta_{22}) & (\beta_{11}\beta_{23}+\beta_{13}\beta_{21}) & (\beta_{11}\beta_{22}+\beta_{12}\beta_{21}) \end{bmatrix} \tag{2.44}$$

Engineering Strain Components

In elasticity of anisotropic solids, it has been a common practice to use engineering strain components in lieu of the strain tensor. The **engineering strain components** are related to the tensorial strain components as

$$\begin{aligned} \varepsilon_{xx} &= e_{xx} \ , \ \varepsilon_{yy} = e_{yy} \ , \ \varepsilon_{zz} = e_{zz} \\ \gamma_{yz} &= 2e_{yz} \ , \ \gamma_{xz} = 2e_{xz} \ , \ \gamma_{xy} = 2e_{xy} \end{aligned} \tag{2.45}$$

in which the (x, y, z) coordinate system is used. Collectively, the engineering strain components do not form a second order tensor, and, thus, do not follow the tensorial coordinate transformation law. However, since they are related to the components of the strain tensor, their coordinate transformation can be performed via the components of the strain tensor.

If the engineering strain components are used, we define

$$\{\varepsilon\} \triangleq \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \triangleq \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

By using the transformation law (2.43) and then (2.45), the coordinate transformation for the engineering strain components is obtained. We have

$$\{\varepsilon'\} = [T_\varepsilon]\{\varepsilon\} \tag{2.46}$$

where

$$[T_\varepsilon] = \begin{array}{c} \\ \\ \\ \\ \\ xy \end{array} \begin{bmatrix} \beta_{11}^2 & \beta_{12}^2 & \beta_{13}^2 & \overset{Y}{\beta_{12}}\overset{z}{\beta_{13}} & \beta_{11}\beta_{13} & \beta_{11}\beta_{12} \\ \beta_{21}^2 & \beta_{22}^2 & \beta_{23}^2 & \beta_{22}\beta_{23} & \beta_{21}\beta_{23} & \beta_{21}\beta_{22} \\ \beta_{31}^2 & \beta_{32}^2 & \beta_{33}^2 & \beta_{32}\beta_{23} & \beta_{31}\beta_{33} & \beta_{31}\beta_{32} \\ 2\beta_{21}\beta_{31} & 2\beta_{22}\beta_{32} & 2\beta_{23}\beta_{33} & (\beta_{22}\beta_{33}+\beta_{23}\beta_{32}) & (\beta_{23}\beta_{31}+\beta_{21}\beta_{33}) & (\beta_{31}\beta_{22}+\beta_{21}\beta_{32}) \\ 2\beta_{11}\beta_{31} & 2\beta_{12}\beta_{32} & 2\beta_{13}\beta_{33} & (\beta_{13}\beta_{32}+\beta_{12}\beta_{33}) & (\beta_{11}\beta_{33}+\beta_{13}\beta_{31}) & (\beta_{11}\beta_{32}+\beta_{12}\beta_{31}) \\ 2\beta_{11}\beta_{21} & 2\beta_{12}\beta_{22} & 2\beta_{13}\beta_{23} & (\beta_{12}\beta_{23}+\beta_{13}\beta_{22}) & (\beta_{11}\beta_{23}+\beta_{13}\beta_{21}) & (\beta_{11}\beta_{22}+\beta_{12}\beta_{21}) \end{bmatrix} \tag{2.47}$$

Note that $[T_\varepsilon] \neq [T_e]$.

Consider the primed coordinate system that is obtained from rotating the original system counterclockwise about the z-axis with a θ angle. Between these two coordinate systems, the coordinate transformation matrix , β_{ij}, is given by

$$\begin{array}{lll} \beta_{11} = \cos\theta & \beta_{12} = \sin\theta & \beta_{13} = 0 \\ \beta_{21} = -\sin\theta & \beta_{22} = \cos\theta & \beta_{23} = 0 \\ \beta_{31} = 0 & \beta_{32} = 0 & \beta_{33} = 1 \end{array} \tag{2.48}$$

Stress and Equilibrium

A stress vector $\mathbf{t}$ is defined as the intensity of force acting on a small surface. Mathematically it is given as

$$\mathbf{t} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{F}}{\Delta S} \tag{2.49}$$

where $\Delta\mathbf{F}$ is the resultant force acting on the area ΔS (see Fig. 2.5).

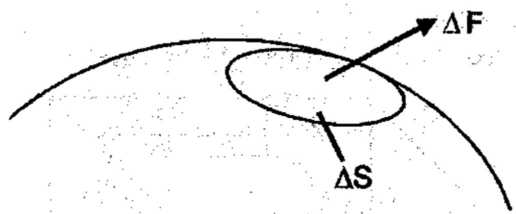


Figure 5

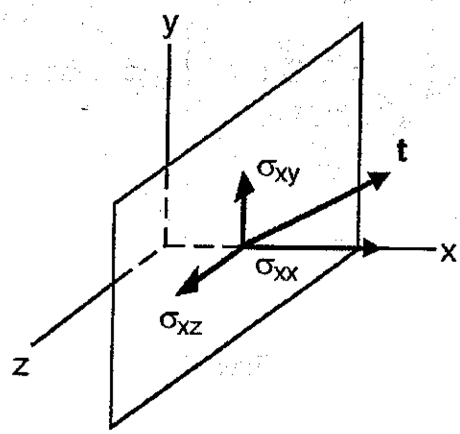


Figure 6

Consider a small surface element that is perpendicular to the x-axis as shown in Fig. 2.6. The stress vector acting on the surface can be decomposed into three components σ_{xx}, σ_{xy} and σ_{xz}, which are parallel to the coordinate axes, respectively. The component σ_{xx} is called the normal stress, and σ_{xy} and σ_{yz} the shear stresses on the x-face. Similar stress components σ_{yy}, σ_{yx}, σ_{yz} and σ_{zz}, σ_{zx} σ_{zy} can be introduced for the y-face and z-face, respectively.

Take a free body in the form of a small rectangular prism with the stresses on the six faces shown in Fig. 2.7. Figure 2.8 shows a side view of this element. The body force **b** (force per unit volume) is represented by the three components b_x, b_y and b_z. If the body is in a state of static or dynamic equilibrium, the resultant moment and force on the element must vanish. It is quite straightforward to show that the moment condition can be satisfied if

$$\sigma_{xy} = \sigma_{yx} \quad , \quad \sigma_{xz} = \sigma_{zx} \quad , \quad \sigma_{yz} = \sigma_{zy}$$

and that the balance of forces can be assured if the stresses satisfy the following differential equations:

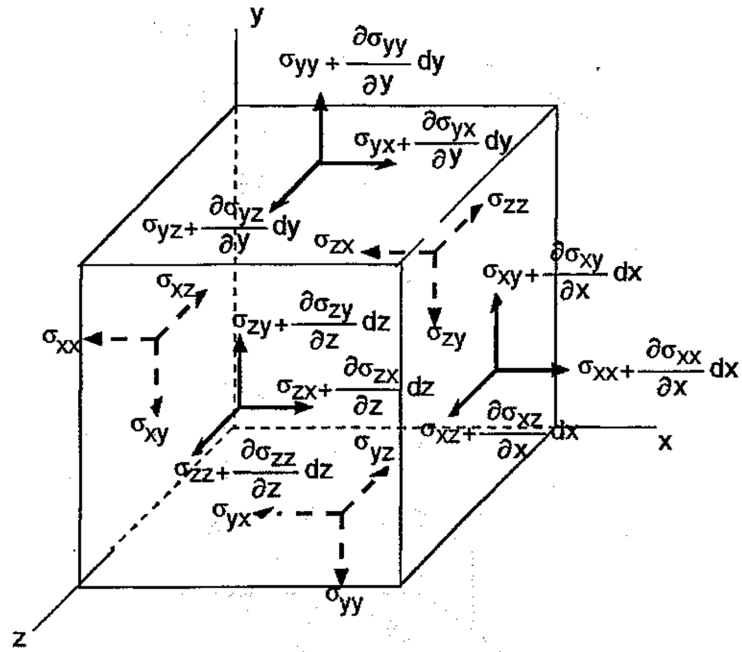


Figure 7

$$
\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x &= 0 \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y &= 0 \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= 0
\end{aligned}
\tag{2.50}
$$

To show that the stress components σ_{xx}, σ_{yy}, $\sigma_{zz}, \sigma_{yz} = \sigma_{zy}, \sigma_{xz} = \sigma_{zx}$, and $\sigma_{xy} = \sigma_{yx}$ are sufficient to describe the state of stress at a point, consider the tetrahedron shown in Fig. 2.9. On the three faces perpendicular to the coordinate axes. The components of the three stress vectors are denoted by σ_{ij}. The stress vector acting on the inclined surface ABC is $\mathbf{t}$, and the unit normal vector is $\mathbf{n}$. The equilibrium of the tetrahedron requires that the resultant force acting on it must vanish leading to the following relations.

$$t_i = \sigma_{ij} n_j \tag{2.51}$$

From the quotation rule and (2.51), we note that σ_{ij} is a second order tensor. Thus, we conclude that the state of stress in a body is completely given by the stress

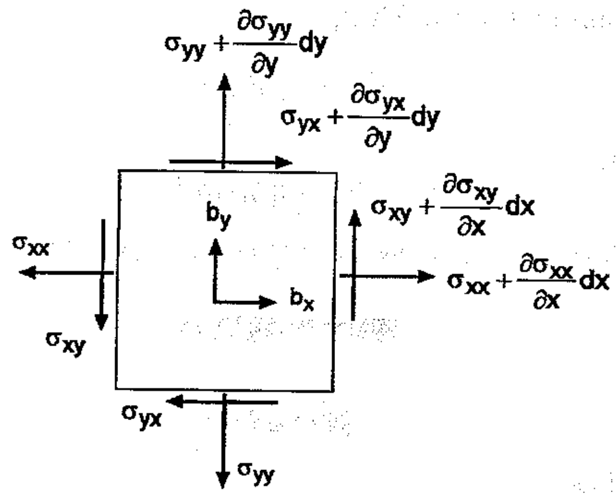


Figure 8

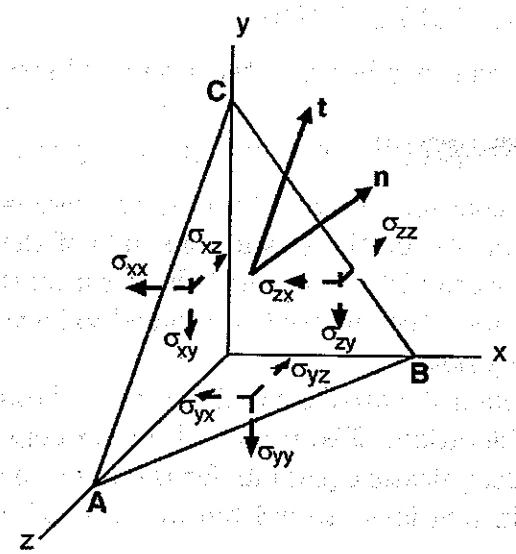


Figure 9

components σ_{ij}. That is, given any surface with the unit normal vector n_i, one is able to determine the stress vector if the stress components are given.

Since σ_{ij} is a second order tensor, its coordinate transformation law is identical to that for the strain tensor e_{ij}. Thus,

$$\sigma'_{ij} = \beta_{im}\beta_{jn}\sigma_{mn} \tag{2.52a}$$

or, in matrix form,

$$[\sigma'] = [\beta]\,[\sigma]\,[\beta]^T \tag{2.52b}$$

If the contracted form of stress (see (2.55)) is used, then (2.52b) can be rewritten as

$$\{\sigma'\} = [T_\sigma]\,\{\sigma\} \tag{2.53}$$

where

$$[T_\sigma] = [T_e] \tag{2.54}$$

and $\{\sigma\}$ is defined as

$$\{\sigma\} \triangleq \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \triangleq \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{2.55}$$

2.3 STRESS-STRAIN RELATIONS

In linear elasticity, the relations between the stress and strain are usually expressed in the form

$$\{\sigma\} = [C]\,\{\varepsilon\} \quad \text{or} \quad \sigma_i = C_{ij}\varepsilon_j \ , \quad i,\, j = 1, ..., 6 \tag{2.56}$$

where the summation convention over the repeated indexes is used, and $[C]$ is a 6×6 matrix whose elements are the elastic constants of the material. The relations given by (2.56) do not distinguish the tensile behavior from the compressive behavior. That is, when (2.56) is used, the material is assumed to have the same stiffness under tension as under compression.

The matrix of **elastic constants**, $[C]$, can be shown to be symmetric from the strain energy consideration. The proof will be presented in Section 2.4. Thus, there exist 21 independent elastic constants for the most anisotropic materials. Note that, if the stress-strain relations are written in terms of components of the strain tensor e_{ij}, i.e.,

$$\{\sigma\} = [c]\{e\} \tag{2.57}$$

then $[c]$ is not symmetric for general anisotropic materials. This may explain why engineering strains have often been used for anisotropic solids.

As the strain energy in a linear elastic material must be positive, the matrix $[C]$ can be shown to be positive definite and thus invertible. We have

$$\{\varepsilon\} = [C]^{-1}\{\sigma\} = [S]\{\sigma\} \tag{2.58}$$

where $[S]$ is called the matrix of **elastic compliances**. The symmetric property of $[C]$ is inherited by $[S]$.

The stress-strain relations in the (x', y', z') system can be obtained by using (2.46), (2.53), (2.56), and (2.58). We have

$$\{\sigma'\} = [T_\sigma][C][T_\varepsilon]^{-1}\{\varepsilon'\} = [C']\{\varepsilon'\} \tag{2.59}$$

$$\{\varepsilon'\} = [T_\varepsilon][S][T_\sigma]^{-1}\{\sigma'\} = [S']\{\sigma'\} \tag{2.60}$$

where $[C']$ and $[S']$ are the elastic constants and compliance matrices, respectively, in the primed coordinate system.

In Section 2.4, the following relations are shown to be valid from the consideration of strain energy density.

$$[T_\sigma]^{-1} = [T_\varepsilon]^T \tag{2.61}$$
$$[T_\varepsilon]^{-1} = [T_\sigma]^T \tag{2.62}$$

Thus. the relation between $\{\sigma'\}$and $\{\varepsilon'\}$ are also expressed in the form

$$[C'] = [T_\sigma]\,[C]\,[T_\sigma]^T \tag{2.63}$$
$$[S'] = [T_\varepsilon]\,[S]\,[T_\varepsilon]^T \tag{2.64}$$

Engineering Moduli

It is a common practice to express elastic material properties in terms of the so called **engineering moduli** which can be measured through the use of simple tension and simple shear tests in which only a single stress component is present, The Young's modulus is defined as the slope of the normal stress - normal strain curve produced in simple tension; the Poisson's ratio is the ratio of the lateral strain and the longitudinal strain in simple tension; and the shear modulus is the slope of the shear stress - shear strain curve produced by a simple shear.

For anisotropic materials, a uniaxial tension may produce shear strains in addition to normal strains, and a shear stress applied in the $x-y$ plane may induce shear strains in the $y-z$ and $x-z$ planes. These extension-shear and shear-shear couplings require additional moduli beyond those normally used for isotropic materials.

For generally anisotropic linearly elastic materials, the compliance matrix can be expressed in engineering moduli as

$$[S]=\begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & \frac{\eta_{yz,x}}{G_{yz}} & \frac{\eta_{xz,x}}{G_{xz}} & \frac{\eta_{xy,x}}{G_{xy}} \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & \frac{\eta_{yz,y}}{G_{yz}} & \frac{\eta_{xz,y}}{G_{xz}} & \frac{\eta_{xy,y}}{G_{xy}} \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & \frac{\eta_{yz.z}}{G_{yz}} & \frac{\eta_{xz,z}}{G_{xz}} & \frac{\eta_{xy,z}}{G_{xy}} \\
\frac{\eta_{x,yz}}{E_x} & \frac{\eta_{y,yz}}{E_y} & \frac{\eta_{z,yz}}{E_z} & \frac{1}{G_{yz}} & \frac{\mu_{xz,yz}}{G_{xz}} & \frac{\mu_{xy,yz}}{G_{xy}} \\
\frac{\eta_{x,xz}}{E_x} & \frac{\eta_{y,xz}}{E_y} & \frac{\eta_{z,xz}}{E_z} & \frac{\mu_{yz,xz}}{G_{yz}} & \frac{1}{G_{xz}} & \frac{\mu_{xy,xz}}{G_{xy}} \\
\frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{\eta_{z,xy}}{E_z} & \frac{\mu_{yz,xy}}{G_{yz}} & \frac{\mu_{xz,xy}}{G_{xz}} & \frac{1}{G_{xy}}
\end{bmatrix} \tag{2.65}$$

where

E_i = Young's modulus in the i-direction

G_{ij} = shear modulus in the i-j plane

ν_{ij} = Poisson's ratio measuring contraction in the j-direction due to uniaxial loading in the i-direction

$\eta_{ij,k}$ = coefficient of mutual influence of the first kind which characterizes normal strain in the k-direction caused by shear in the i-j plane

$\eta_{k,ji}$ = coefficient of mutual influence of the first kind which characterizes shear the i-j plane caused by normal stress in the k direction

$\mu_{ij,kl}$ = Chentsov's coefficients which characterize shear strain in the k-l plane caused by shear stress in the i-j plane

Material Symmetries

If the internal composition of a material possesses symmetry of any kind, then symmetry can be observed in its elastic properties. The presence of symmetry further reduces the number of independent elastic constants.

Let (x_1, x_2, x_3) be a coordinate system and (x_1', x_2', x_3') be the second system which is symmetric to the first in accordance with the form of its elastic symmetry. Since both systems are equivalent with respect to elastic properties, the stress-strain relations should be identical in both coordinate systems. In other words, the matrix $[C]$ should remain unchanged in the equivalent coordinate systems.

Monoclinic Material

To illustrate the invariant property of $[C]$, let us consider a material with one elastic symmetry plane, say, the $x_1 - x_2$ plane. The equivalent coordinate system

(x_1', x_2', x_3') arranged as shown in Fig. 2.10 is obtained from the (x_1, x_2, x_3) system by reversing the direction of the x_3-axis. The base vectors for the unprimed and primed systems are

$$\mathbf{e}_1 = (1, 0, 0) \ , \ \ \mathbf{e}_2 = (0, 1, 0) \ , \ \ \mathbf{e}_3 = (0, 0, 1)$$

and

$$\mathbf{e}_1' = (1, 0, 0) \ , \ \ \mathbf{e}_2' = (0, 1, 0) \ , \ \ \mathbf{e}_3' = (0, 0, -1)$$

respectively. The corresponding transformation matrix is given by

$$[\beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

coordinate tranformation matrix

Noting that $[T_\sigma] = [T_e]$, the transformation matrices $[T_\sigma]$ and $[T_\varepsilon]$ can be calculated using (2.44) and (2.47), respectively. With this $[T_\sigma]$, the stress components in the primed system are related to those in the unprimed system as

$$\begin{Bmatrix} \sigma_{11}' \\ \sigma_{22}' \\ \sigma_{33}' \\ \sigma_{23}' \\ \sigma_{13}' \\ \sigma_{12}' \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -\sigma_{23} \\ -\sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{2.66}$$

For the strain components, we have

$$\begin{Bmatrix} \varepsilon_{11}' \\ \varepsilon_{22}' \\ \varepsilon_{33}' \\ \gamma_{23}' \\ \gamma_{13}' \\ \gamma_{12}' \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ -\gamma_{23} \\ -\gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{2.67}$$

The elastic symmetry requires that

$$\{\sigma'\} = [C]\{\varepsilon'\} \tag{2.68}$$

The first equation in (2.68) is

$$\sigma_{11}' = C_{11}\varepsilon_{11}' + C_{12}\varepsilon_{22}' + C_{13}\varepsilon_{33}' + C_{14}\gamma_{23}' + C_{15}\gamma_{13}' + C_{16}\gamma_{12}' \tag{2.69}$$

By using the relations given by (2.67-2.68), the above equation becomes

$$\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} - C_{14}\gamma_{23} - C_{15}\gamma_{13} + C_{16}\gamma_{12} \tag{2.70}$$

Same sign.

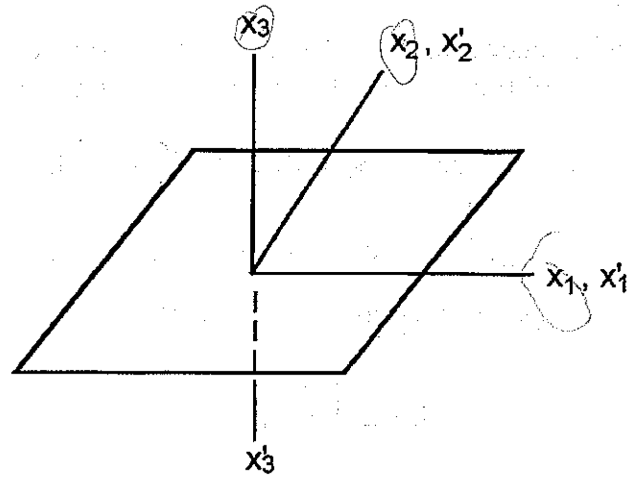


Figure 10

To satisfy the invariant property of $[C]$ in these two coordinate systems, it is necessary that

$$C_{14} = C_{15} = 0$$

Similar examinations of other equations in (2.69) lead to

$$C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0$$

The matrix of elastic constants then reduces to the form:

13

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \tag{2.71}$$

Note that the number of independent elastic constants reduces to 13 as a result of the presence of a plane of elastic symmetry.

Orthotropic Material

For orthotropic materials, there are three mutually orthogonal planes of material symmetry. Let x_1, x_2, and x_3 be the three axes of a Cartesian coordinate system which are perpendicular to the three planes of symmetry, respectively. Due to this symmetry, no change in the matrix $[C]$ will occur when any of the axes is reversed in direction. Thus, if a new coordinate system obtained by reversing at least one of the coordinate axes is used to describe the stresses and strains, then the stress-strain relations should remain unchanged; i.e., the matrix $[C]$ should be invariant with respect to such changes in the coordinate system. This condition on $[C]$ requires certain relationships among the elastic constants as discussed earlier. Explicitly, this condition leads to the following

$$
\begin{aligned}
C_{14} &= C_{15} = C_{16} = C_{24} = C_{25} = C_{26} = C_{34} \\
&= C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0
\end{aligned}
$$

Using the coordinate system (x_1, x_2, x_3), the matrix of elastic constants assumes the form

(a)

$$
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\tag{2.72}
$$

It is noted that the number of independent moduli reduces to nine. In terms of the engineering elastic moduli, the stress-strain relations for an orthotropic material can be expressed in the form

S

fiber in 0° direction.

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}
=
\begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}}
\end{bmatrix}
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}
\tag{2.73}
$$

Due to the symmetric property of the compliance matrix, the following relationships are obtained

Orthotropic material:

$$
E_1\nu_{21} = E_2\nu_{12}\ , \quad E_2\nu_{32} = E_3\nu_{23}\ , \quad E_3\nu_{13} = E_1\nu_{31}
\tag{2.74}
$$

$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$

The relations between the stiffnesses C_{ij} and the compliances S_{ij} are

$$
\begin{aligned}
C_{11} &= \frac{S_{22}S_{33} - S_{23}^2}{\Delta} \\
C_{22} &= \frac{S_{11}S_{33} - S_{13}^2}{\Delta} \\
C_{33} &= \frac{S_{11}S_{22} - S_{12}^2}{\Delta} \\
C_{12} &= \frac{S_{13}S_{23} - S_{12}S_{33}}{\Delta} \\
C_{13} &= \frac{S_{12}S_{23} - S_{13}S_{22}}{\Delta} \\
C_{23} &= \frac{S_{12}S_{13} - S_{23}S_{11}}{\Delta} \\
C_{44} &= \frac{1}{S_{44}} \ , \ C_{55} = \frac{1}{S_{55}} \ , \ C_{66} = \frac{1}{S_{66}}
\end{aligned}
\tag{2.75}
$$

where

$$
\Delta = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{vmatrix} = S_{11}S_{22}S_{33} + 2S_{12}S_{23}S_{13} - S_{13}^2S_{22} - S_{11}S_{23}^2 - S_{12}^2S_{33}
$$

The expressions for S_{ij} in terms of C_{ij} are similar to (2.76) with S_{ij} and C_{ij} interchanged. From (2.74), it is evident that the engineering moduli can be expressed in terms of the compliances as

Orthotropic Material

$$
\begin{aligned}
E_1 &= \frac{1}{S_{11}} \ , \ E_2 = \frac{1}{S_{22}} \ , \ E_3 = \frac{1}{S_{33}} \ , \\
\nu_{12} &= -\frac{S_{21}}{S_{11}} \ , \ \nu_{13} = -\frac{S_{31}}{S_{11}} \ , \ \nu_{23} = -\frac{S_{32}}{S_{22}} \ , \\
G_{23} &= \frac{1}{S_{44}} \ , \ G_{13} = \frac{1}{S_{55}} \ , \ G_{12} = \frac{1}{S_{66}}
\end{aligned}
\tag{2.76}
$$

The elastic constants C_{ij} can also be expressed in terms of the engineering moduli as

$$
\begin{aligned}
C_{11} &= \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta'} \\
C_{22} &= \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta'} \\
C_{33} &= \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta'} \\
C_{12} &= \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta'} \\
C_{13} &= \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta'} \\
C_{23} &= \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta'} \\
C_{44} &= G_{23}\ ,\ \ C_{55} = G_{13}\ ,\ \ C_{66} = G_{12}
\end{aligned}
\tag{2.77}
$$

where

$$\Delta' = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

The expression for the engineering moduli in terms of elastic constants C_{ij} can be obtained from (2.77) by using the relations between S_{ij} and C_{ij}. An alternate approach is to consider simple tension and shear deformations and obtain the moduli from their respective definitions. The results are

? $E_2 = E_1$

$$
\begin{aligned}
E_1 &= C_{11} + \frac{2C_{12}C_{13}C_{23} - C_{12}^2C_{33} - C_{13}^2C_{22}}{C_{22}C_{33} - C_{23}^2} \\
\nu_{12} &= \frac{C_{12}C_{33} - C_{13}C_{23}}{C_{22}C_{33} - C_{23}^2}
\end{aligned}
\tag{2.78}
$$

$G_{12} = C_{66}$

The other Young's moduli (E_2 and E_3) and Poisson's ratios (ν_{23} and ν_{13}) can be obtained by permuting the indices. The relations between the shear moduli and the elastic constants are obvious from (2.78).

Table 2.1 lists the engineering moduli and thermal expansion coefficients of some typical advanced composite materials.

Table 2.1 Elastic Moduli of Unidirectional Composites

Composite Type	E_1 GPa (msi)	E_2 GPa (msi)	G_{12} GPa (msi)	ν_{12}	α_1 $^o/C$	α_2 $^o/C$
S2 Glass/Epoxy	43.3 (6.2)	12.7 (1.8)	4.5 (0.65)	0.29	5.0	26.0
AS4/3501-6 (Carbon/Epoxy)	140 (20)	10.0 (1.45)	7.0 (1.0)	0.30	-0.9	27.0
Kevlar-49/Epoxy	87 (12.5)	5.5(0.8)	2.2 (0.31)	0.34	-2.0	60.0
Boron/Epoxy	200 (29)	21 (3.0)	5.4 (0.78)	0.17	6.0	30.0
Boron/Aluminum	235 (33.6)	135 (19.3)	47 (6.8)	0.30	6.0	20.0

Transversely Isotropic Material

An axis of material symmetry is defined as an axis with respect to which the material has identical properties. Thus, any two material segments having symmetrical positions with respect to this axis have the same stiffness. If the axis of symmetry is parallel to the x_1-axis, then the x_2- and x_3-axes can be directed in any directions (except that they should remain perpendicular to each other) without altering the value of $[C]$. The x_2-x_3 plane is usually referred to as an isotropic plane.

A transversely isotropic solid is a solid which has an axis of symmetry perpendicular to a plane of symmetry. The stress-strain relations should remain invariant with respect to a rotation of the x_2- and x_3-axes about the x_1-axis. As a result of this invariant condition, $[C]$ reduces to

5

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22}-C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{2.79}$$

The number of independent elastic moduli is five.

Transversely isotropic solids can be considered as a subset of orthotropic materials. If an orthotropic material also possesses transverse isotropy, then its engineering moduli have the following relations

$$\begin{aligned} E_2 &= E_3 \ , \ \nu_{12} = \nu_{13} \ , \ G_{12} = G_{13} \\ G_{23} &= \frac{E_2}{2(1+\nu_{23})} \end{aligned}$$

Many unidirectional fiber composites can be modeled as transversely isotropic solids with reasonable accuracy.

Isotropic Material

In an isotropic material, every plane is a plane of symmetry and every direction is an axis of symmetry. The matrix $[C]$ reduces to

$$\sigma \qquad \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \underbrace{\tfrac{1}{2}(C_{11}-C_{12})}_{G_{xy}} \end{bmatrix} \qquad \varepsilon \tag{2.80}$$

where

$$\begin{aligned} C_{11} &= \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \\ C_{12} &= \frac{\nu E}{(1+\nu)(1-2\nu)} \end{aligned}$$

It is evident that two elastic constants are sufficient to describe the stress-strain relationship in an isotropic material. It is also noted that the shear modulus

$$G = \frac{1}{2}(C_{11} - C_{12}) = \frac{E}{2(1+\nu)}$$

is not independent of the Young's modulus E and Poisson's ratio ν.

2.4 STRAIN ENERGY

It is assumed that there exists a strain energy function $W(\varepsilon_i)$ such that

$$\sigma_i = \frac{\partial W}{\partial \varepsilon_i} \qquad i = 1, 2, ..., 6.$$

If the material is linearly elastic, i.e., $\sigma_i = C_{ij}\varepsilon_j$, where C_{ij} are constants, then

$$\frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = C_{ij} \tag{2.81}$$

The symmetric property of $[C]$ is thus obvious.

The strain energy function for linearly elastic materials can be expressed in terms of the strain components as

$$W = \frac{1}{2} C_{ij}\varepsilon_i\varepsilon_j = \frac{1}{2}\{\varepsilon\}^T [C] \{\varepsilon\} = \frac{1}{2}\{\varepsilon\}^T \{\sigma\} \tag{2.82}$$

where the superscript T denotes the transposed matrix.

Alternatively, the strain energy function can be expressed in terms of stress components and the compliances S_{ij} as

$$W = \frac{1}{2} S_{ij}\sigma_i\sigma_j = \frac{1}{2}\{\sigma\}^T [S] \{\sigma\} = \frac{1}{2}\{\sigma\}^T \{\varepsilon\} \tag{2.83}$$

Since the strain energy function is a scalar and is invariant with respect to coordinate transformation, we can write

$$W = \frac{1}{2}\{\varepsilon'\}^T \{\sigma'\} \tag{2.84}$$

or

$$W = \frac{1}{2}\{\sigma'\}^T \{\varepsilon'\} \tag{2.85}$$

Using coordinate transformations for stress and strain, we obtain the following equations from (2.84-2.87).

$$\begin{aligned} W &= \frac{1}{2}\{\varepsilon\}^T [T_\sigma]^{-1} \{\sigma'\} \\ &= \frac{1}{2}\{\sigma\}^T [T_\varepsilon]^{-1} \{\varepsilon'\} \\ &= \frac{1}{2}\{\varepsilon\}^T [T_\varepsilon]^T \{\sigma'\} \\ &= \frac{1}{2}\{\sigma\}^T [T_\sigma]^T \{\varepsilon'\} \end{aligned}$$

The relations of (2.61) and (2.62) are thus obvious.

Since the strain energy density W is always positive, from (2.83) and (2.84), we conclude that both $[C]$ and $[S]$ are positive. Thus, their diagonal terms C_{11}, C_{22}, ..., C_{66}, and S_{11}, S_{22}, ..., S_{66} are positive quantities, and the determinants of all the principal diagonal submatrices of both matrices are also positive.

Consider orthotropic materials. From (2.76) and the fact that $\Delta > 0$, we obtain

$$\begin{aligned} S_{22}S_{33} &> S_{23}^2 \\ S_{11}S_{33} &> S_{13}^2 \\ S_{11}S_{22} &> S_{12}^2 \end{aligned} \tag{2.86}$$

These inequalities can be expressed in terms of the engineering moduli (see (2.74)) as:

$$\begin{aligned} \frac{E_3}{E_2} &> \nu_{32}^2 \\ \frac{E_1}{E_3} &> \nu_{13}^2 \\ \frac{E_1}{E_2} &> \nu_{12}^2 \end{aligned} \tag{2.87}$$

Similar inequalities involving ν_{23}, ν_{31}, and ν_{21} can be obtained from (2.75) and (2.89) as

$$\begin{aligned}\frac{E_2}{E_3} &> \nu_{23}^2\\ \frac{E_3}{E_1} &> \nu_{31}^2\\ \frac{E_2}{E_1} &> \nu_{21}^2\end{aligned} \tag{2.88}$$

Using the conditions $\Delta > 0$ or $\Delta' > 0$, we obtain the additional inequality on the engineering moduli:

$$2\,\nu_{21}\nu_{32}\nu_{13} < 1 - \left(\frac{E_1}{E_2}\right)\nu_{21}^2 - \left(\frac{E_2}{E_3}\right)\nu_{32}^2 - \left(\frac{E_3}{E_1}\right)\nu_{13}^2 \tag{2.89}$$

or equivalently,

$$2\,\nu_{21}\nu_{32}\nu_{13} < 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} \tag{2.90}$$

2.5 DISPLACEMENT-EQUATIONS OF MOTION

When an elastic body is subjected to dynamic loading, wave motion or vibration is set off. Including the inertia force in the body force in (2.50), we obtain the equations of motion. In the dynamic analysis of solids, it is often more convenient to write the equations of motion in terms of displacement components, u_1, u_2, and u_3. By using the stress-strain relations and the strain-displacement relations, the displacement equations of motion can easily be obtained from (2.50). For an orthotropic solid, the displacement equations of motion with respect to the material principal axes are

$$\begin{aligned}
&C_{44}\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\frac{\partial^2 u_1}{\partial x_2^2} + C_{55}\frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial}{\partial x_1}\left[(C_{11} - C_{44})\frac{\partial u_1}{\partial x_1}\right.\\
&\qquad \left. +(C_{12} + C_{66})\frac{\partial u_2}{\partial x_2} + (C_{13} + C_{55})\frac{\partial u_3}{\partial x_3}\right] + b_1 = \rho\frac{\partial^2 u_1}{\partial t^2}\\[2ex]
&C_{66}\frac{\partial^2 u_2}{\partial x_1^2} + C_{55}\frac{\partial^2 u_2}{\partial x_2^2} + C_{44}\frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial}{\partial x_2}\left[(C_{12} + C_{66})\frac{\partial u_1}{\partial x_1}\right.\\
&\qquad \left. +(C_{22} - C_{55})\frac{\partial u_2}{\partial x_2} + (C_{23} + C_{44})\frac{\partial u_3}{\partial x_3}\right] + b_2 = \rho\frac{\partial^2 u_2}{\partial t^2}\\[2ex]
&C_{55}\frac{\partial^2 u_3}{\partial x_1^2} + C_{44}\frac{\partial^2 u_3}{\partial x_2^2} + C_{66}\frac{\partial^2 u_3}{\partial x_3^2} + \frac{\partial}{\partial x_3}\left[(C_{13} + C_{55})\frac{\partial u_1}{\partial x_1}\right.\\
&\qquad \left. +(C_{23} + C_{44})\frac{\partial u_2}{\partial x_2} + (C_{33} - C_{66})\frac{\partial u_3}{\partial x_3}\right] + b_3 = \rho\frac{\partial^2 u_3}{\partial t^2}
\end{aligned} \tag{2.91}$$

In (2.93), ρ is the mass density; b_1, b_2, and b_3 are the components of the body force (excluding the inertia force); and u_1, u_2, and u_3 are the displacement components in the material principal directions, x_1, x_2, and x_3, respectively.

PROBLEMS

2.1. Verify that $t_i \delta_{ij} = t_j$ when δ_{ij} is the Kronecker delta, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

2.2. The base vectors for two Cartesian coordinate systems are given by

$$\mathbf{e}_1 = (1,0,0) \quad , \quad \mathbf{e}_2 = (0,1,0) \quad , \quad \mathbf{e}_3 = (0,0,1)$$

and

$$\mathbf{e}_1' = (0,0,1) \quad , \quad \mathbf{e}_2' = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \quad , \quad \mathbf{e}_3' = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

respectively. Find the transformation matrix β_{ij} which gives

$$x' = \beta_{ij} x_j$$

2.3. Verify that $\beta_{ik} \beta_{jk} = \delta_{ij}$. Find the components of the vector $\boldsymbol{v} = \mathbf{e}_1 + \mathbf{2e}_2 + \mathbf{3e}_3$ in the primed coordinate system defined in Prob 2.2.

2.4. Consider the two coordinate systems in Prob 2.2. Given the stress tensor σ_{ij} in the unprimed system as

$$[\sigma] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

find the stress components σ'_{ij} in the primed coordinate system.

If the components of a second order tensor in the primed system are given by

$$[t'] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

find the stress components t_{ij} in the unprimed system.

2.5. Given a 3rd order tensor T_{ijk}:

$$T_{111} = T_{222} = T_{333} = 2$$
$$T_{123} = T_{231} = T_{312} = 1$$

All other components $= 0$.

Find the components T'_{ijk} in the primed system. Use the two coordinate systems in Prob 2.2.

2.6. Use the relation $ds^2 - ds_0^2 = 2e_{ij}dx_idx_j$ in solving the following problems.

a) Show that the elongation of any material element in any direction is a constant if

$$[e_{ij}] = \begin{bmatrix} e_0 & 0 & 0 \\ 0 & e_0 & 0 \\ 0 & 0 & e_0 \end{bmatrix}$$

b) The deformation in a body is given by the strain tensor

$$[e_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-2}$$

Find the elongation (per unit length) of a material element parallel to the direction $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$.

2.7. For infinitesimal strain components e_{ij}, show that e_{ii} represents the volume change (per unit volume).

2.8. The state of stress in a body is uniform and is given by

$$\begin{aligned} \sigma_{11} &= 4 \text{ MPa} \quad , \quad \sigma_{12} = 2 \text{ MPa} \quad , \quad \sigma_{13} = 0 \text{ MPa} \\ \sigma_{22} &= 3 \text{ MPa} \quad , \quad \sigma_{23} = 0 \text{ MPa} \quad , \quad \sigma_{33} = 2 \text{ MPa} \end{aligned}$$

Find the three components of the stress vector $\underline{t}$ on the surface ABCD as shown in Figure 2.11. Find the normal component σ_n (perpendicular to the surface) of the stress vector.

2.9. A state of hydrostatic stress is given by

$$[\sigma] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}$$

$\sigma_{ij} n_j = \sigma_0 (\delta_{ij} n_j) = \hat{t}$

$\sigma_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$j=1$: $\delta_{11} n_1 = n_i$, $\delta_{21} n_1 = 0$

$j=2$: $\delta_{12} n_2 = 0$, $\delta_{22} n_2 = n_2$

Show that on any surface the force (or stress vector) is always perpendicular to the surface and that the magnitude of the stress vector is equal to σ_0.

$\hat{t}_i = \sigma_0 \hat{n}_i$ normal vector

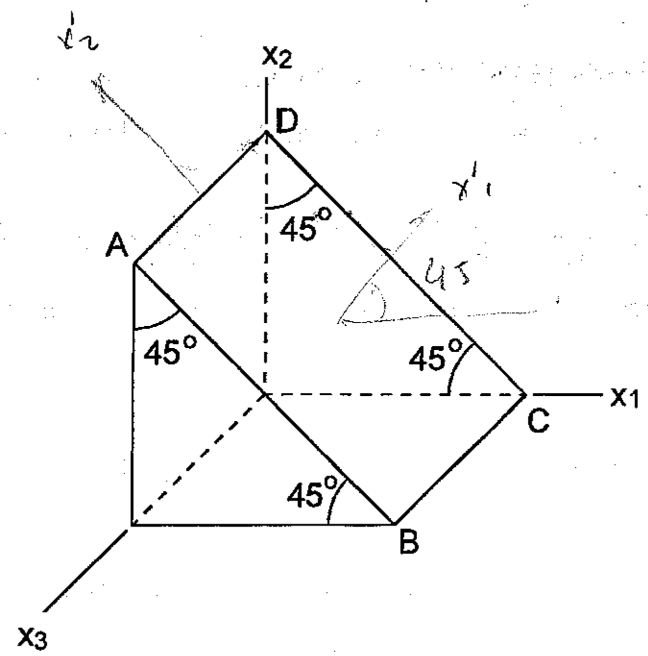


Figure 11

2.10. Consider a 2-D isotropic solid. The stress-strain relations can be expressed as

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E} & -\dfrac{\nu}{E} & 0 \\ -\dfrac{\nu}{E} & \dfrac{1}{E} & 0 \\ 0 & 0 & \dfrac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Works for different coordinate systems

changing coordinate system

Show that $G = \dfrac{E}{2(1+\nu)}$ by using the invariant property of the stress-strain relations with respect to coordinate transformation.

2.11. Show that the range of Poisson's ratios for 3-D isotropic solids is $-1 \leq \nu \leq 1/2$. why

2.12. Consider an orthotropic solid with engineering moduli

$$\begin{aligned} &E_1 = 140\,GPa \quad &, \quad &E_2 = E_3 = 10\,GPa \\ &G_{12} = G_{13} = 7\,GPa \quad &, \quad &G_{23} = 3.36\,GPa \\ &\nu_{12} = \nu_{13} = 0.3 \quad &, \quad &\nu_{23} = 0.49 \end{aligned}$$

Find the corresponding elastic constants C_{11}, C_{22}, and C_{33}. Also consider the case with $\nu_{12} = \nu_{13} = \nu_{23} = 0$. Compare the values of E_1, E_2, and E_3 with those of C_{11}, C_{22}, and C_{33}, respectively.

2.13. A principal direction n_i of the stress tensor σ_{ij} satisfies

$$\sigma_{ij} n_j = \sigma n_i$$

where σ is the associated principal stress. Show that the principal directions of σ_{ij} and the corresponding strain tensor e_{ij} coincide in isotropic solids but not in anisotropic solids.

Chapter 3
ANALYSIS OF A LAMINA

3.1 PLANE STRESS EQUATIONS FOR ORTHOTROPIC MATERIALS

Many structural applications of fiber-reinforced composite materials are in the form of thin layers or laminates, and a state of plane stress parallel to the laminate can be assumed with reasonable accuracy. For this reason, formulations in plane stress are of particular interest.

For a state of plane stress parallel to the $x_1 - x_2$ plane in an orthotropic solid (i.e., $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$), Eq. (2.74) reduces to

$$\left\{ \begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{array} \right\} = \begin{bmatrix} \dfrac{1}{E_1} = S_{11} & -\dfrac{\nu_{12}}{E_1} = S_{12} & 0 = S_{16} \\[2ex] -\dfrac{\nu_{12}}{E_1} = S_{21} & \dfrac{1}{E_2} = S_{22} & 0 = S_{26} \\[2ex] 0 = S_{61} & 0 = S_{62} & \dfrac{1}{G_{12}} = S_{66} \end{bmatrix} \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\} \tag{3.1}$$

in which the relations given by (2.75) have been used. Note that there are four independent elastic constants involved.

Inverting (3.1) we obtain

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\} = \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\[2ex] \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\[2ex] 0 & 0 & G_{12} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{array} \right\} \tag{3.2}$$

The 3×3 matrix in the above relationship is usually denoted by

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \tag{3.3}$$

in which the elements Q_{ij} are called reduced stiffnesses which should not be confused with elastic constants C_{ij}.

In stress analyses, sometimes a coordinate system x-y is set up which does not always coincide with the material principal axes, x_1 and x_2 as illustrated in Fig. 3.1. The two sets of stress components with respect to these two coordinates systems are related by the reduced transformation matrix $[T_\sigma]$:

$$\boxed{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = [T_\sigma] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}} \tag{3.4}$$

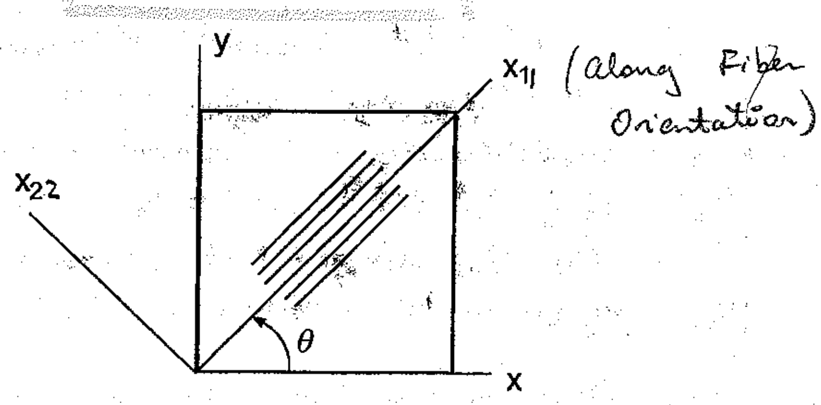


Fig 3.1

where

$$[T_\sigma] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{3.5}$$

The elements of $[T_\sigma]$ can be obtained from (2.44) by noting $[T_\sigma] = [T_e]$

In the same manner, the strains with respect to the two coordinate systems are related by

$$\boxed{\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = [T_\varepsilon] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}} \tag{3.6}$$

where

$$[T_\varepsilon] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{3.7}$$

Note that the inverses $[T_\sigma]^{-1}$ and $[T_\varepsilon]^{-1}$ can be obtained by replacing θ in (3.5) and (3.7) with -θ, respectively.

Using the transformation matrices $[T_\sigma]$ and $[T_\varepsilon]$, we have

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [T_\sigma]^{-1} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = [T_\sigma]^{-1} [Q] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = [T_\sigma]^{-1} [Q] [T_\varepsilon] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

Thus, the stress-strain relations for the state of plane stress parallel to $x-y$ (x_1-x_2) plane become

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\}=\left[\bar{Q}\right]\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} \tag{3.8}$$

where

$$\left[\bar{Q}\right]=[T_\sigma]^{-1}[Q][T_\varepsilon]=[T_\varepsilon]^T[Q][T_\varepsilon] \tag{3.9}$$

The explicit expressions for the elements in $[Q]$ are given by

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta+2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta+Q_{22}\sin^4\theta\\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta+Q_{12}\left(\sin^4\theta+\cos^4\theta\right)\\
\bar{Q}_{22} &= Q_{11}\sin^4\theta+2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta+Q_{22}\cos^4\theta\\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta+(Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta\\
\bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta+(Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta\\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta+Q_{66}\left(\sin^4\theta+\cos^4\theta\right)
\end{aligned} \tag{3.10}$$

The fact that $\left[\bar{Q}\right]$ is a full matrix indicates that the in-plane shear deformation γ_{xy} is coupled with the normal deformations ε_{xx} and ε_{yy}.

By following a similar procedure, we obtain

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\}=\left[\bar{S}\right]\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\} \tag{3.11}$$

where

$$\left[\bar{S}\right]=[T_\sigma]^T[S][T_\sigma] \tag{3.12}$$

and

$$\begin{aligned}
\bar{S}_{11} &= S_{11}\cos^4\theta+(2S_{12}+S_{66})\sin^2\theta\cos^2\theta+S_{22}\sin^4\theta\\
\bar{S}_{12} &= S_{12}\left(\sin^4\theta+\cos^4\theta\right)+(S_{11}+S_{22}-S_{66})\sin^2\theta\cos^2\theta\\
\bar{S}_{22} &= S_{11}\sin^4\theta+(2S_{12}+S_{66})\sin^2\theta\cos^2\theta+S_{22}\cos^4\theta\\
\bar{S}_{16} &= (2S_{11}-2S_{12}-S_{66})\sin\theta\cos^3\theta+(2S_{12}-2S_{22}+S_{66})\sin^3\theta\cos\theta\\
\bar{S}_{26} &= (2S_{11}-2S_{12}-S_{66})\sin^3\theta\cos\theta+(2S_{12}-2S_{22}+S_{66})\sin\theta\cos^3\theta\\
\bar{S}_{66} &= 2\left(2S_{11}+2S_{22}-4S_{12}-S_{66}\right)\sin^2\theta\cos^2\theta+S_{66}\left(\sin^4\theta+\cos^4\theta\right)
\end{aligned} \tag{3.13}$$

The stress-strain relations in an arbitrary (x,y) coordinate system can also be expressed in **apparent engineering moduli** as

$-\frac{\nu_{xy}}{E_x}\sigma_{xx} + \frac{1}{E_y}\sigma_{yy} = 0$

$\sigma_{yy} = -\frac{\nu_{xy} E_y \sigma_x}{E_1}$

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} = \left[\begin{array}{ccc}\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}}\\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}}\\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}}\end{array}\right]\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\} \tag{3.14}$$

For an orthotropic material, the apparent engineering moduli can be expressed in terms of the principal engineering elastic constants through the use of (3.14), (3.13), and (3.1). The relations are

ν_{12}

$$\begin{aligned}
\frac{1}{E_x} &= \frac{1}{E_1}\cos^4\theta + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\sin^2\theta\cos^2\theta + \frac{1}{E_2}\sin^4\theta \\
\nu_{xy} &= E_x\left[\frac{\nu_{12}}{E_1} - \left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta\right] \\
\frac{1}{E_y} &= \frac{1}{E_1}\sin^4\theta + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\sin^2\theta\cos^2\theta + \frac{1}{E_2}\cos^4\theta \qquad -\frac{\nu_{12}}{E_2} \\
\frac{1}{G_{xy}} &= \frac{1}{G_{12}} + 4\left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta \\
\eta_{x,xy} &= E_x\left[\left(\frac{2}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin\theta\cos^3\theta - \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^3\theta\cos\theta\right] \\
\eta_{y,xy} &= E_y\left[\left(\frac{2}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^3\theta\cos\theta - \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin\theta\cos^3\theta\right]
\end{aligned} \tag{3.15}$$

Variations of the apparent moduli against fiber orientation θ for some composites are given in Fig. 3.2. Note that for the composites considered in Fig. 3.2, the maximum coupling between extension and shear occurs between $\theta = 10°$ and $20°$.

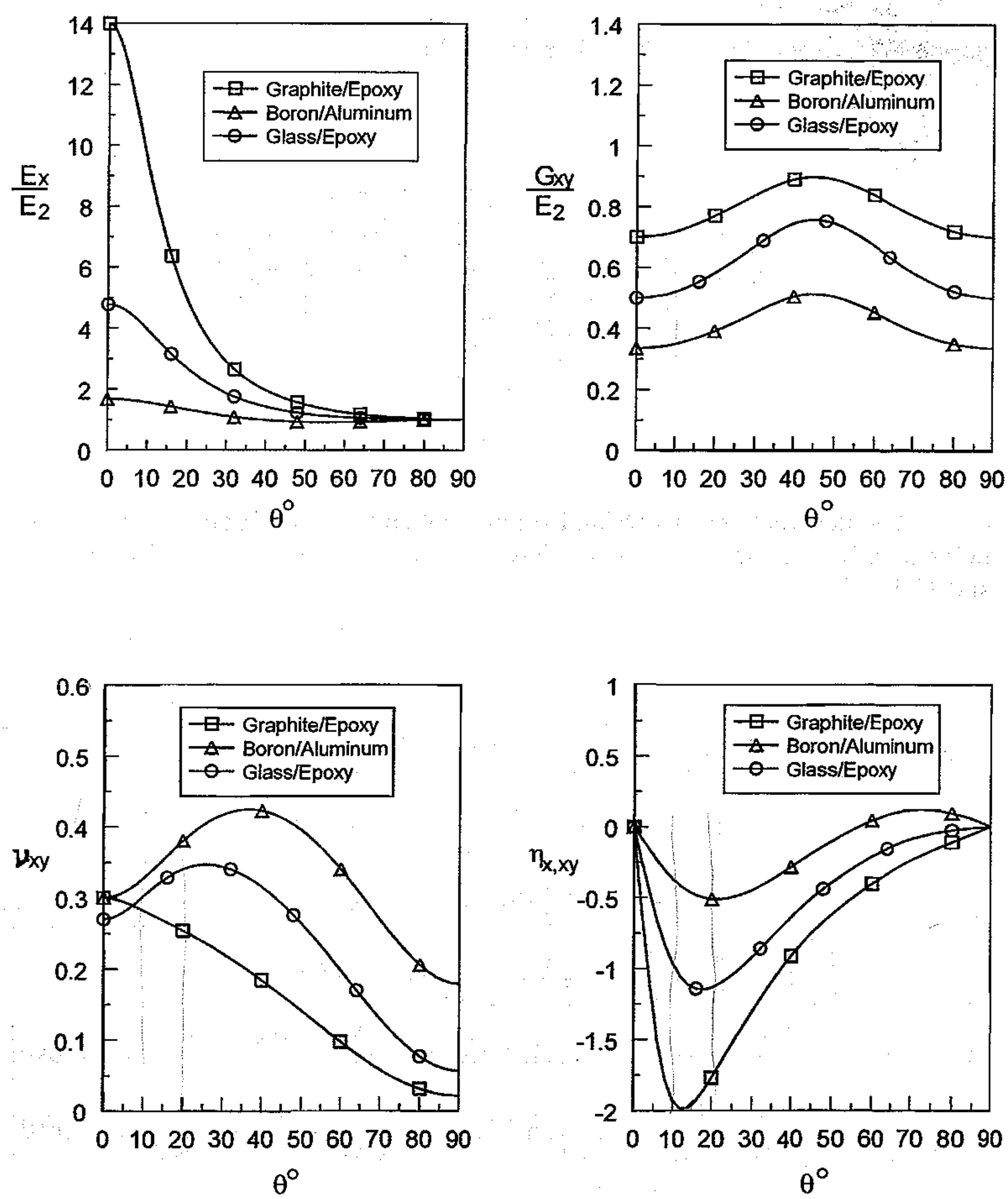

Graphite/Epoxy
Boron/Aluminum
Glass/Epoxy
Ex/E2
Gxy/E2
νxy
ηx,xy
θ°
0 10 20 30 40 50 60 70 80 90

Fig 3.2

3.2 INVARIANTS

By substituting the following trigonometric identities

$$
\begin{aligned}
\cos^4\theta &= \frac{1}{8}(3+4\cos 2\theta+\cos 4\theta) \\
\cos^3\theta\sin\theta &= \frac{1}{8}(2\sin 2\theta+\sin 4\theta) \\
\cos^2\theta\sin^2\theta &= \frac{1}{8}(1-\cos 4\theta) \\
\cos\theta\sin^3\theta &= \frac{1}{8}(2\sin 2\theta-\sin 4\theta) \\
\sin^4\theta &= \frac{1}{8}(3-4\cos 2\theta+\cos 4\theta)
\end{aligned}
\tag{3.16}
$$

into (3.10), the transformed $\bar{Q}_{ij}$ can be rewritten as

$$
\begin{aligned}
\bar{Q}_{11} &= U_1+U_2\cos 2\theta+U_3\cos 4\theta \\
\bar{Q}_{22} &= U_1-U_2\cos 2\theta+U_3\cos 4\theta \\
\bar{Q}_{12} &= U_4-U_3\cos 4\theta \\
\bar{Q}_{16} &= \frac{1}{2}U_2\sin 2\theta+U_3\sin 4\theta \\
\bar{Q}_{26} &= \frac{1}{2}U_2\sin 2\theta-U_3\sin 4\theta \\
\bar{Q}_{66} &= U_5-U_3\cos 4\theta
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
U_1 &= \frac{1}{8}(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66}) \\
U_2 &= \frac{1}{2}(Q_{11}-Q_{22}) \\
U_3 &= \frac{1}{8}(Q_{11}+Q_{22}-2Q_{12}-4Q_{66}) \\
U_4 &= \frac{1}{8}(Q_{11}+Q_{22}+6Q_{12}-4Q_{66}) \\
U_5 &= \frac{1}{8}(Q_{11}+Q_{22}-2Q_{12}+4Q_{66})
\end{aligned}
\tag{3.18}
$$

are independent of coordinate transformation. From (3.18) we note that

$$
\begin{aligned}
I_1 &= \bar{Q}_{11}+\bar{Q}_{22}+2\bar{Q}_{12}=Q_{11}+Q_{22}+2Q_{12} \\
I_2 &= \bar{Q}_{66}-\bar{Q}_{12}=Q_{66}-Q_{12}
\end{aligned}
\tag{3.19}
$$

are two "invariants"; i.e., their relations with Q_{ij} and $\overline{Q}_{ij}$ are not affected by a coordinate rotation in the x_1-x_2 plane. By combining (3.18) and (3.19), we obtain

$$\begin{aligned} U_1 &= \frac{1}{8}(3I_1 + 4I_2) \\ U_4 &= \frac{1}{8}(I_1 - 4I_2) \\ U_5 &= \frac{1}{8}(I_1 + 4I_2) \end{aligned}$$

However, among the three only two are independent as

$$U_5 = \frac{1}{2}(U_1 - U_4)$$

These invariants were first introduced by Tsai and Pagano [3.1]. In the form of (3.17), the transformed stiffnesses $\bar{Q}_{ij}$ depend only on four invariants U_1, U_2, U_3 and U_4 and the fiber orientation θ, while in the form of (3.10), $\bar{Q}_{ij}$ depend on six constants Q_{ij}. The expressions of (3.17) also identify the quantities that vary with θ.

In a similar manner, the transformed compliances $\bar{S}_{ij}$ can be written as

$$\begin{aligned} \bar{S}_{11} &= U_1' + U_2'\cos 2\theta + U_3'\cos 4\theta \\ \bar{S}_{22} &= U_1' - U_2'\cos 2\theta + U_3'\cos 4\theta \\ \bar{S}_{12} &= U_4' - U_3'\cos 4\theta \\ \bar{S}_{16} &= U_2'\sin 2\theta + 2U_3'\sin 4\theta \\ \bar{S}_{26} &= U_2'\sin 2\theta - 2U_3'\sin 4\theta \\ \bar{S}_{66} &= 4U_5' - 4U_3'\cos 4\theta \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} U_1' &= \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}) \\ U_2' &= \frac{1}{2}(S_{11} - S_{22}) \\ U_3' &= \frac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}) \\ U_4' &= \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66}) \\ U_5' &= \frac{1}{8}(S_{11} + S_{22} - 2S_{12} + S_{66}) \end{aligned} \tag{3.21}$$

3.3 OFF-AXIS LOADING

Consider a state of uniform deformation in a composite panel produced by applying a uniaxial stress $\sigma_{xx} = \sigma_0$ in the x-direction, see Fig. 3.3. The uniform state of deformation is given by the strains

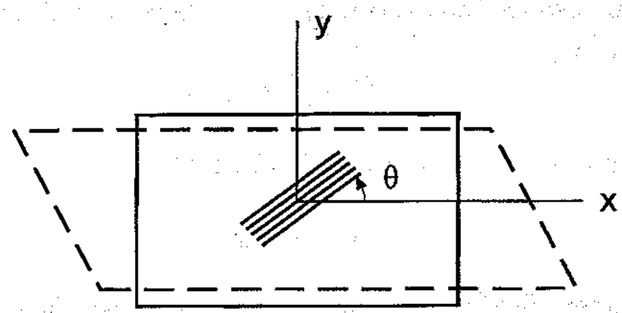


Fig 3.3

$$
\begin{aligned}
\varepsilon_{xx} &= \bar{S}_{11}\sigma_0 \\
\varepsilon_{yy} &= \bar{S}_{12}\sigma_0 \\
\gamma_{xy} &= \bar{S}_{16}\sigma_0
\end{aligned} \tag{3.22}
$$

It is seen that shear deformation can result from application of a normal load, except when x- and y-axes coincide with the material principal axes, x_1 and x_2. Coupling between normal deformation and shear deformation does not exist in isotropic solids.

Integrating the strain-displacement relations for ε_{xx} and ε_{yy} yields the displacement components:

$$u_x = \varepsilon_{xx}x + f(y) \tag{3.23}$$

$$u_y = \varepsilon_{yy}y + g(x) \tag{3.24}$$

where $f(y)$ and $g(x)$ are arbitrary functions. Substituting (3.24) and (3.25) into

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \tag{3.25}$$

we obtain

$$\gamma_{xy} = f'(y) + g'(x) = \bar{S}_{16}\sigma_0 \tag{3.26}$$

where a prime indicates differentiation with respect to the argument. From (3.26), it is obvious that $f(y)$ and $g(x)$ must be linear functions of y and x, respectively, i.e.,

$$f(y) = C_1 y + C_3 \tag{3.27}$$

$$g(x) = C_2 x + C_4 \tag{3.28}$$

Thus, the displacements (3.24) and (3.25) can be expressed as

$$u_x = \bar{S}_{11}\sigma_0 x + C_1 y + C_3 \tag{3.29}$$

$$u_y = \bar{S}_{12}\sigma_0 y + C_2 x + C_4 \tag{3.30}$$

$(S_{11}\sigma_0 + C_2)x + (S_{12}\sigma_0 + C_1)y + \overset{=0}{C_3 + C_4} = 0$

$S_{11}\sigma_0 = -C_2 \; ; \; S_{12}\sigma_0 = -C_1$

Removing the rigid body translations, we set $C_3 = C_4 = 0$. To suppress the rigid body rotation, we assume that the horizontal edges of the panel remain horizontal after deformation, i.e.,

$$\frac{\partial u_y}{\partial x} = C_2 = 0 \tag{3.31}$$

The remaining constant C_1 is obtained from (3.26) as

$$C_1 = \bar{S}_{16}\sigma_0 \tag{3.32}$$

Thus, the displacement field in the composite panel under the uniform stress $\sigma_{xx} = \sigma_0$ is

$$u_x = \bar{S}_{11}\sigma_0 x + \bar{S}_{16}\sigma_0 y \tag{3.33}$$
$$u_y = \bar{S}_{12}\sigma_0 y \tag{3.34}$$

For the AS4/3501-6 graphite/epoxy composite, the elastic moduli are

$$\begin{aligned} E_1 &= 140 \text{ GPa}(20 \times 10^6\text{psi}) \\ E_2 &= 10 \text{ GPa}(1.45 \times 10^6\text{psi}) \\ G_{12} &= 6.9 \text{ GPa}(1.0 \times 10^6\text{psi}) \\ \nu_{12} &= 0.3 \end{aligned}$$

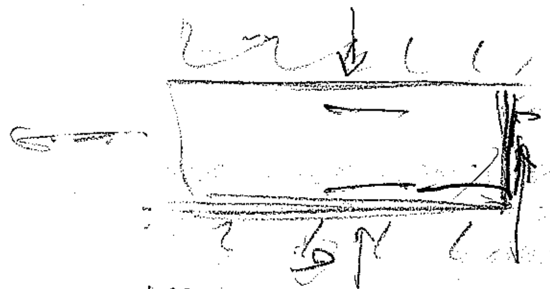
If the off-axis angle θ is 45°, then we have

$$\begin{aligned} \bar{S}_{11} &= 0.615 \times 10^{-10} \text{ m}^2/\text{N} \\ \bar{S}_{12} &= -0.245 \times 10^{-10} \text{ m}^2/\text{N} \\ \bar{S}_{16} &= -0.47 \times 10^{-10} \text{ m}^2/\text{N} \end{aligned} \tag{3.35}$$

The deformed shape of the panel can be determined from the displacement field (3.33-3.34), which is depicted in Fig. 3.3.

3.4 A BEAM THEORY FOR ANALYSIS OF OFF-AXIS SPECIMENS

Off-axis coupon specimens are often tested to determine composite material properties. As discussed in the previous section, the off-axis specimen under uniaxial loading tends to deform into a skew parallelogram in the plane of the specimen. Normal specimen gripping arrangements prevent this deformation. Suppression of the end rotations induces bending moments in the plane of the specimen, and, thus, a uniform state of stress cannot be achieved. In this section, a beam theory is developed for the purpose of analyzing off-axis composite coupon specimens subjected to various end conditions.



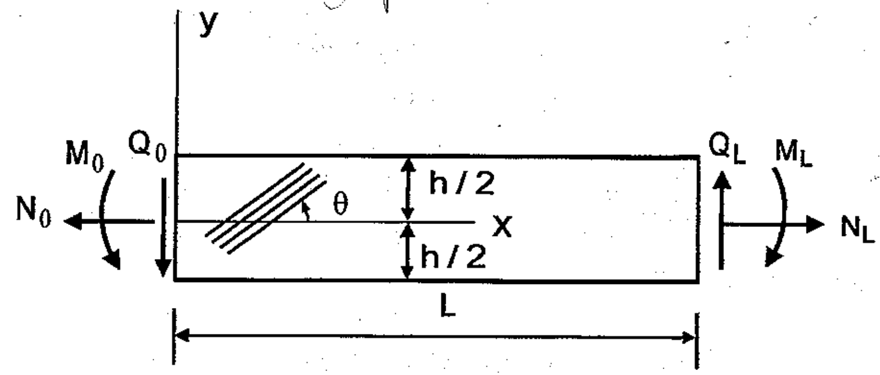


Fig 3.4

Consider a beam-like off-axis composite specimen as shown in Fig. 3.4. Without loss of generality, the width (in the z-direction) is taken as unity. A state of plane stress parallel to the x-y plane is assumed. The tractions on the longitudinal edges are absent. We assume that $\sigma_{yy} = 0$ everywhere. Thus,

$$\sigma_{yy} = \bar{Q}_{12}\varepsilon_{xx} + \bar{Q}_{22}\varepsilon_{yy} + \bar{Q}_{26}\gamma_{xy} = 0 \tag{3.36}$$

Using (3.36) to eliminate ε_{yy} from the stress-strain relations (3.8), we obtain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}^*_{11} & \bar{Q}^*_{16} \\ \bar{Q}^*_{16} & \bar{Q}^*_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \end{Bmatrix} \tag{3.37}$$

where

$$\begin{aligned} \bar{Q}^*_{11} &= \bar{Q}_{11} - \frac{\bar{Q}^2_{12}}{\bar{Q}_{22}} \\ \bar{Q}^*_{16} &= \bar{Q}_{16} - \frac{\bar{Q}_{26}\bar{Q}_{12}}{\bar{Q}_{22}} \\ \bar{Q}^*_{66} &= \bar{Q}_{66} - \frac{\bar{Q}^2_{26}}{\bar{Q}_{22}} \end{aligned} \tag{3.38}$$

The displacements u_x and u_y are, in general, functions of x and y. As an approximation, we assume the following expansions:

$$u_x(x,y) = u_0(x) + y\psi(x) \tag{3.39}$$
$$u_y(x,y) = v_0(x) \tag{3.40}$$

where u_0 and v_0 are the horizontal and vertical displacements at the mid-plane of the beam, and ψ is the rotation of the cross-section. The strains corresponding to the approximate beam displacements are

$$\varepsilon_{xx} = \varepsilon_0(x) + y\kappa(x) \tag{3.41}$$

$$\gamma_{xy} = \gamma_0(x) \tag{3.42}$$

where

$$\varepsilon_0(x) = \frac{du_0}{dx}\ ,\quad \kappa(x) = \frac{d\psi}{dx}\ ,\quad \gamma_0(x) = \frac{dv_0}{dx} + \psi \tag{3.43}$$

Substitution of (3.41-3.42) into (3.37) yields

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{xy} \end{array} \right\} = \left[\begin{array}{cc} \bar{Q}^*_{11} & \bar{Q}^*_{16} \\ \bar{Q}^*_{16} & \bar{Q}^*_{66} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_0 \\ \gamma_0 \end{array} \right\} + y \left\{ \begin{array}{c} \kappa \\ 0 \end{array} \right\} \tag{3.44}$$

The resultant extensional force N, shear force Q, and moment M are defined as

$$\begin{aligned} N &= \int_{-h/2}^{h/2} \sigma_{xx} dy = a_{11}\varepsilon_0 + a_{16}\gamma_0 \\ Q &= \int_{-h/2}^{h/2} \sigma_{xy} dy = a_{16}\varepsilon_0 + a_{66}\gamma_0 \\ M &= \int_{-h/2}^{h/2} \sigma_{xx} y dy = D\kappa \end{aligned} \tag{3.45}$$

where

$$\begin{aligned} a_{ij} &= h\bar{Q}^*_{ij} \\ D &= \frac{h^3}{12}\bar{Q}^*_{11} \end{aligned} \tag{3.46}$$

Pa. m.

Assume that loads are applied only at the two ends of the off-axis composite beam as shown in Fig. 3.4. From the consideration of force and moment equilibria, we easily derive the following:

$$N = N_L = N_0 = \text{ constant} \tag{3.47}$$

$$Q = Q_L = Q_0 = \text{ constant} \tag{3.48}$$

$$M = D\frac{d\psi}{dx} = Q_0 x + M_0 \tag{3.49}$$

Simple Tension

SIMPLE TENSION

Consider the case of simple tension with $N = N_0$, $Q_0 = Q_L = 0$, $M_0 = M_L = 0$, and the ends are not restrained. From (3.47) we have

$$
\begin{aligned}
a_{11}\varepsilon_0 + a_{16}\gamma_0 &= N_0 && (3.50)\\
a_{16}\varepsilon_0 + a_{66}\gamma_0 &= 0 && (3.51)\\
D\kappa &= 0 && (3.52)
\end{aligned}
$$

Equation (3.52) yields

$$\psi = \psi_0 = \text{constant} \tag{3.53}$$

Solving (3.50) and (3.51), we obtain

$$
\begin{aligned}
\varepsilon_0 &= \frac{N_0}{a_{11} - \dfrac{a_{16}^2}{a_{66}}} && (3.54)\\
\gamma_0 &= -\frac{a_{16}}{a_{66}}\varepsilon_0 && (3.55)
\end{aligned}
$$

Comparing (3.55) with (3.43), we have

$$\frac{dv_0}{dx} + \psi = -\frac{a_{16}}{a_{66}}\varepsilon_0 \tag{3.56}$$

which leads to

$$v_0 = -\left(\frac{a_{16}}{a_{66}}\varepsilon_0 + \psi\right)x + C \tag{3.57}$$

It is easy to show that

$$C = 0 \quad \text{and} \quad \psi = -\frac{a_{16}}{a_{66}}\varepsilon_0 = -\frac{\bar{Q}^*_{16}}{\bar{Q}^*_{66}}\varepsilon_0 \tag{3.58}$$

in order to satisfy the end conditions $v_0(0) = v_0(L) = 0$. This implies that $v_0 = 0$ everywhere.

Since ε_0 is constant, thus $u_0 = \varepsilon_0 x$, and the horizontal displacement u_x is given by

$$u_x(x, y) = u_0 + y\psi = \varepsilon_0\left(x - \frac{\bar{Q}^*_{16}}{\bar{Q}^*_{66}}y\right) \tag{3.59}$$

Substituting (3.54) into (3.59) together with the definitions of a_{ij}, we obtain

$$\boxed{u_x(x, y) = \sigma_0\left[\frac{\bar{Q}^*_{66}}{\bar{Q}^*_{11}\bar{Q}^*_{66} - \bar{Q}^{*2}_{16}}x - \frac{\bar{Q}^*_{16}}{\bar{Q}^*_{11}\bar{Q}^*_{66} - \bar{Q}^{*2}_{16}}y\right]} \tag{3.60}$$

where $\sigma_0 = N_0/h$ is the uniform applied stress. It can be shown that the solutions given by (3.33) and (3.60) are identical.

Rigidly Gripped Ends

In testing off-axis composite coupon specimens, rigid grips apply a tensile force $N = N_0$ with the end conditions

$$\psi = 0 \qquad at \quad x = 0,\ L \tag{3.61}$$

$$v_0 = 0 \qquad at\ x = 0,\ L \tag{3.62}$$

Integrate (3.49) to obtain

$$D\psi = \frac{1}{2}Q_0x^2 + M_0x + C \tag{3.63}$$

The end conditions (3.61) give

$$C = 0 \qquad \text{and} \qquad M_0 = -\frac{1}{2}LQ_0$$

Thus,

$$D\psi = \frac{1}{2}Q_0x^2 - \frac{1}{2}LQ_0x \tag{3.64}$$

Elimination of ε_0 using the first two equations in (3.45) yields

$$\gamma_0 = \frac{dv_0}{dx} + \psi = \frac{b_{11}}{h}Q_0 - \frac{b_{16}}{h}N_0 \tag{3.65}$$

where

$$\begin{aligned} b_{11} &= \frac{a_{11}h}{a_{11}a_{66} - a_{16}^2} = \frac{\bar{Q}_{11}^*}{\bar{Q}_{11}^*\bar{Q}_{66}^* - \bar{Q}_{16}^{*2}} \quad \frac{1}{P_c} \\ b_{16} &= \frac{a_{16}h}{a_{11}a_{66} - a_{16}^2} = \frac{\bar{Q}_{16}^*}{\bar{Q}_{11}^*\bar{Q}_{66}^* - \bar{Q}_{16}^{*2}} = \frac{1}{P_c} \end{aligned} \tag{3.66}$$

Substituting ψ from (3.64) into (3.65) yields

$$\frac{dv_0}{dx} = \left(-\frac{1}{2D}x^2 + \frac{L}{2D}x + \frac{b_{11}}{h}\right)Q_0 - \frac{b_{16}}{h}N_0$$

which is readily integrated with the result

$$v_0 = \left(-\frac{1}{6D}x^3 + \frac{L}{4D}x^2 + \frac{b_{11}}{h}x\right)Q_0 - \frac{b_{16}}{h}N_0x \tag{3.67}$$

in which the integration constant is set equal to zero because of the boundary conditions $v_0 = 0$ at $x = 0$.The other condition ($v_0 = 0$ at $x = L$) leads to

$$Q_0 = \frac{b_{16}}{b_{11} + \dfrac{L^2 h}{12D}} N_0 = \frac{b_{16}\bar{Q}_{11}^*}{b_{11}\bar{Q}_{11}^* + \dfrac{L^2}{h^2}} N_0 \tag{3.68}$$

Substituting (3.68) into (3.67), we obtain the beam deflection as

$$v_0 = \frac{b_{16}N_0}{b_{11}\bar{Q}_{11}^* + \left(\dfrac{L}{h}\right)^2} \left[-2\left(\frac{x}{h}\right)^3 + 3\left(\frac{L}{h}\right)\left(\frac{x}{h}\right)^2 - \left(\frac{L}{h}\right)^2\left(\frac{x}{h}\right) \right] \tag{3.69}$$

Note that the deflection is S-shaped as depicted in Fig. 3.5, and that $v_0 = 0$ at $x = L/2$.

Fig. 3. 5

The induced bending moments at the two ends are

$$M_0 = -M_L = -\frac{1}{2} L Q_0 = -\frac{b_{16}\bar{Q}_{11}^* L\, N_0}{2\left(b_{11}\bar{Q}_{11}^* + \dfrac{L^2}{h^2}\right)} \tag{3.70}$$

Solutions of (3.68) and (3.70) indicate that the induced shear force and bending moment would diminish as the length (L) becomes much larger than the depth (h) of the beam.

Using two-dimensional plane stress elasticity theory, Pagano and Halpin [3.2] solved a similar problem but with different end conditions to approximate clamped off-axis specimens. Instead of satisfying the clamped end condition, their solution satisfies the approximate boundary conditions,

$$u_x = 0, \qquad u_y = \frac{\partial u_x}{\partial y} = 0$$

at $x = 0,\ y = 0$ and $x = L,\ y = 0$.

3.5 OBLIQUE TAB FOR OFF-AXIS TESTING

As discussed in the previous section, tension testing of an off-axis composite coupon specimen using rigid grips would induce bending, and a simple uniaxial stress field cannot be obtained. Conceptually, the simple tension stress field could be realized if the grip could allow the associated shear strain to take place freely. Such a grip is not yet available. Recently, Sun and Chung [3.3] developed an oblique end tab that can generate a state of uniform tension in off-axis specimens using rigid grips.

Consider an off-axis coupon specimen under the uniaxial stress $\sigma_{xx} = \sigma_0$. The displacement u_x is given by (3.33) as

$$u_x = \bar{S}_{11}\sigma_0 x + \bar{S}_{16}\sigma_0 y$$

An alternative expression of (3.33) is given by (3.60). From the above equation, we see that the positions (x, y) that have the same longitudinal displacement must satisfy the following equation.

$$\left(\bar{S}_{11}x + \bar{S}_{16}y\right)\sigma_0 = \text{constant} \tag{3.71}$$

Equation (3.71) represents a straight line making an angle ϕ (see Fig. 3.6)against the x-axis with

$$\cot\phi = -\frac{\bar{S}_{16}}{\bar{S}_{11}} \tag{3.72}$$

If (3.59) is used, then we obtain an alternative expression as

$$\cot\phi = \frac{\bar{Q}^*_{16}}{\bar{Q}^*_{66}} \tag{3.73}$$

For $\theta = 0°$ and $90°$, $\bar{S}_{16} = \bar{Q}^*_{16} = 0$ and $\phi = 90°$.

The plot of ϕ versus fiber orientation θ for the AS4/3501-6 graphite/epoxy composite is given in Fig. 3.7. In view of this unique property, an end tab can be designed so that its end is oblique, making an angle ϕ with respect to the x-axis as shown in Fig. 3.6. If made more rigid than the specimen, this end tab would produce a uniform longitudinal displacement along its oblique edge that conforms with the displacement field produced by uniform tension.

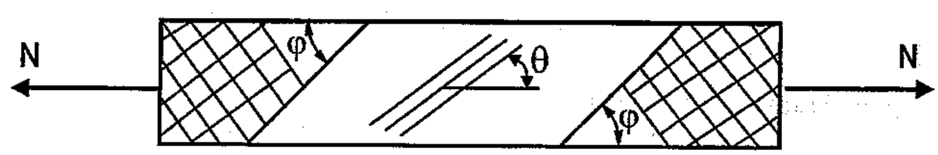


Fig. 3.6

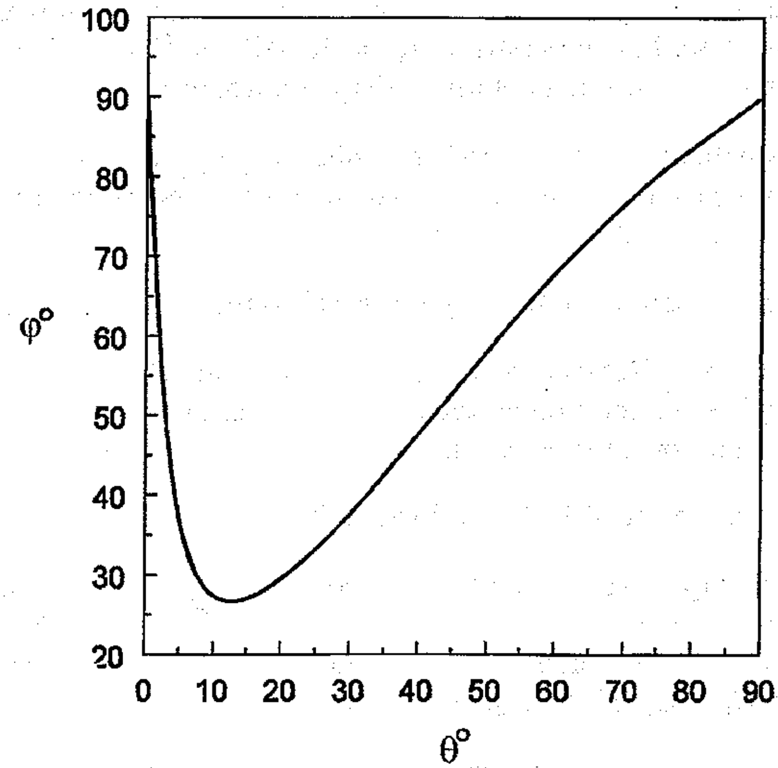


Fig 3.7

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3.2 Pagano, N.J. and Halpin, J.C., "Influence of End Constraint in the Testing of Anisotropic Bodies," Journal of Composite Materials, Vol. 2, No. 1, 1968, pp. 18-31.

3.3 Sun, C.T. and Chung, I., "An Oblique End Tab Design for Testing Off-Axis Composite Specimens," Composites, Vol. 24, No. 8, 1993, pp. 619-623.

PROBLEMS

3.1 Given a carbon/epoxy composite panel under uniaxial loading, i.e.,$\sigma_{xx} = \sigma_0$, $\sigma_{yy} = \sigma_{xy} = 0$, plot γ_{xy} as a function of the fiber orientation θ. The composite properties are

$$E_1 = 140 \text{ GPa},\ E_2 = 10 \text{ GPa},\ G_{12} = 7 \text{ GPa},\ \nu_{12} = 0.3$$ $\nu_{23} = 0.49$

3.2 Consider a rectangular composite panel with $\theta = 45°$ (material properties are given in Prob 3.1) subjected to $\sigma_{xx} = 10$ MPa, $\sigma_{yy} = 0$, $\sigma_{xy} = \tau$. Find τ that is necessary to keep the deformed shape rectangular.

3.3 Plot the extension-shear coupling coefficients $\eta_{x,xy}$ and $\eta_{xy,x}$ versus θ for the composite given in Prob. 3.1. Find the θ's that correspond to the maximum values of $\eta_{x,xy}$ and $\eta_{xy,x}$, respectively.

3.4 If the carbon/epoxy composite panel is subjected to a shear stress τ, find

1) The fiber orientation at which σ_{11} is maximum.
2) The fiber orientation at which γ_{xy} is minimum.
3) The fiber orientation at which $|\varepsilon_{xx}/\gamma_{xy}|$ is maximum.

Compare the result with that of Prob. 3.3.

3.5 Express the apparent moduli for fiber composites in terms of $\sin n\theta$ and $\cos n\theta$. Use these expressions to find the angle θ (other than 0° or 90°) for which G_{xy} could become a maximum or minimum. State the conditions for both cases. Assume $E_1 > E_2$, $E_1 > G_{12}$, $\nu_{12} = 0.3$.

3.6 For the composite given in Prob 3.1, find the θ's that give the maximum and minimum values of the apparent Poisson's ratio ν_{xy} respectively. If you are allowed to alter the value of G_{12}, find the values of G_{12} that would yield $\nu_{xy} = 0$ and $\nu_{xy} = -0.1$, respectively.

3.7 A rectangular composite panel is confined by two smooth rigid walls as shown in Fig. 3.8. Apply compressive stress $\sigma_{xx} = -\sigma_o$. Find the resulting stress σ_{yy} as a function of fiber orientation θ. Plot σ_{yy}/σ_o versus θ. Assume the composite moduli to be those in Prob 3.1.

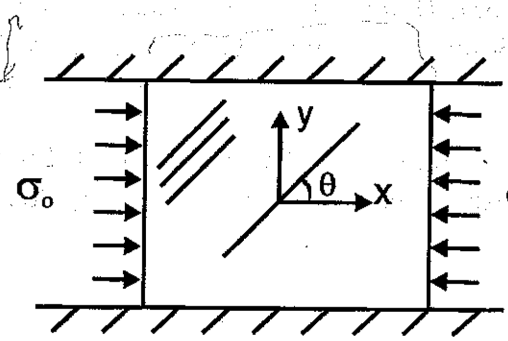
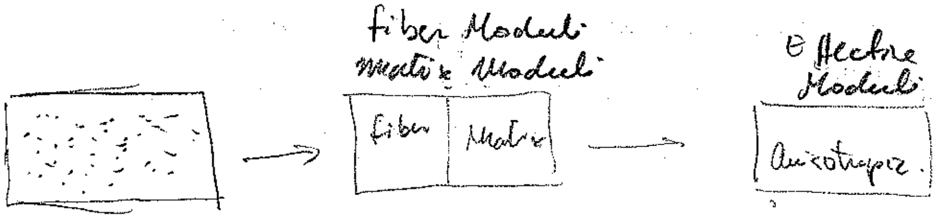


Fig 3.8

3.8 1. In Prob 3.7, find the value of σ_o that is needed to produce $\varepsilon_{xx} = -0.01$ for $\theta = 30°$. If the rigid walls are removed, what is the σ_o?

3.9 Show that the displacements given by (3.33) and (3.60) are identical.

3.10 Plot the end moment of a clamped-clamped off-axis ($\theta = 45°$) composite beam versus its span to depth ratio (L/h). The beam is subjected to a uniaxial stress σ_0. Use the composite properties given by Prob. 3.1.



Chapter 4
EFFECTIVE ELASTIC MODULI

Fiber composites are heterogeneous media with distinct phases of fibers and matrix. Due to the presence of large numbers of fibers, it is impractical to analyze a composite by retaining its distinct phases and their exact geometries. If a composite appears statistically homogeneous at a large scale, then it may be effectively represented by a macro homogeneous solid with certain effective moduli that describe the "average" material properties of the composite. Once these effective moduli are derived, a composite is then analyzed as a homogeneous anisotropic solid.

Many methods have been proposed for evaluating effective moduli of a composite. Some are primitive, such as the so called rule of mixtures, while some attempt to account for the local deformation in the fiber and matrix, and thus are called micro mechanics approaches. Common to these approaches is the consideration of **a representative volume element** (RVE) or a typical unit cell which is a subregion of the composite that repeats itself over the entire body. The effective material properties are obtained from this representative volume element through various assumed deformations. Since in a homogeneous medium the field quantities such as displacement, stress and strain must be described by continuous functions, the state of stress or strain in the RVE's in a small neighborhood must vary slowly. In other words, the characteristic length of the macro deformation must be relatively large as compared with the characteristic dimension of the heterogeneity of the original medium.

4.1 THE EQUIVALENT HOMOGENEOUS SOLID

To construct the equivalent homogeneous medium to represent the microscopically heterogeneous counterpart, one must define the macro stress and macro strain. One definition is derived from averaging the stress tensor and strain tensor over the volume of the RVE, i.e.,

$$\bar{\sigma}_{ij}= \frac{1}{V}\int_V \sigma_{ij}(x,y,z)dV \qquad i,j=1,2,3 \tag{4.1}$$

and

$$\bar{\varepsilon}_{ij}= \frac{1}{V}\int_V \varepsilon_{ij}(x,y,z)dV \qquad i,j=1,2,3 \tag{4.2}$$

respectively, where σ_{ij} and ε_{ij} are the actual stress and strain tensor components in the RVE. In contracted notations, we write

$$\bar{\sigma}_i = \frac{1}{V}\int\limits_V \sigma_i(x,y,z)dV \qquad i = 1,\cdots,6 \tag{4.3}$$

and

$$\bar{\varepsilon}_i = \frac{1}{V}\int\limits_V \varepsilon_i(x,y,z)dV \qquad i = 1,\cdots,6 \tag{4.4}$$

respectively. For a fiber-reinforced composite, the average stress and strain can be written as

$$\bar{\sigma}_i = \frac{1}{V}\left(\int\limits_{V_f} \sigma_i^f(x,y,z)dV + \int\limits_{V_m} \sigma_i^m(x,y,z)dV\right) \tag{4.5}$$

and

$$\bar{\varepsilon}_i = \frac{1}{V}\left(\int\limits_{V_f}^{f} \varepsilon_i(x,y,z)dV + \int\limits_{Vm} \varepsilon_i^m(x,y,z)dV\right) \tag{4.6}$$

respectively. In (4.5,4.6), V_f denotes the region occupied by the fiber, V_m the region occupied by the matrix, and σ_i^f $\left(\varepsilon_i^f\right)$ and σ_i^m (ε_i^m) are the stress (strain) fields in the fiber and matrix regions, respectively.

The **effective elastic constants** C_{ij} are defined from the following average stress-average strain relations:

$$\bar{\sigma}_i = C_{ij}\,\bar{\varepsilon}_j \quad i,\, j = 1,\cdots,6 \tag{4.7}$$

or in matrix notation,

$$\{\bar{\sigma}\} = [C]\{\bar{\varepsilon}\}$$

The inverse relations are given by introducing the effective compliances S_{ij}, i.e.,

$$\bar{\varepsilon}_i = S_{ij}\,\bar{\sigma}_j \tag{4.8}$$

or

$$\{\bar{\varepsilon}\} = [S]\{\bar{\sigma}\}$$

The equivalence between the actual heterogeneous composite medium and the homogeneous medium given by the average stresses, strains, and the effective elastic constants needs to be examined. For this purpose, we consider a macroscopically homogeneous state in which an RVE is subjected to appropriate boundary tractions t_i or boundary displacements u_i that would produce uniform stress $(\bar{\sigma}_{ij})$ and strain $(\bar{e}_{ij})$ in a homogeneous medium, i.e.,

$$u_i\,(S) = \bar{e}_{ij}\,x_j \tag{4.9}$$

or

$$t_i\,(S) = \overline{\sigma}_{ij}\,n_j \tag{4.10}$$

The total strain energy stored in the RVE is equal to the work done at the boundary, i.e.,

$$\begin{aligned}
\frac{1}{2}\int_S t_i u_i dS &= \frac{1}{2}\int_S \overline{\sigma}_{ij}\,n_j\,\overline{e}_{ik}\,x_k dS = \frac{1}{2}\int_V \frac{\partial}{\partial x_j}\left(\overline{\sigma}_{ij}\overline{\varepsilon}_{ik}\,x_k\right) dV \\
&= \frac{1}{2}\,\overline{\sigma}_{ij}\overline{e}_{ik}\int_V \frac{\partial x_k}{\partial x_j} dV \\
&= \frac{V}{2}\,\overline{\sigma}_{ij}\overline{e}_{ik}\cdot\delta_{jk} \\
&= \frac{V}{2}\left(\overline{\sigma}_{ij}\overline{e}_{ij}\right) = \frac{1}{2}V\,\overline{\sigma}_i\overline{\varepsilon}_i
\end{aligned} \tag{4.11}$$

where S is the bounding surface of the RVE.

If the same boundary conditions are applied to the original heterogeneous RVE, then

$$\frac{1}{2}\int_S t_i u_i dS = \frac{1}{2}\int_S \sigma_{ij} n_j u_i dS \tag{4.12}$$

where σ_{ij} represents the actual stress field in the RVE, and the relation $t_i = \sigma_{ij}\,n_j$ has been used.

Using Gauss theorem [4.1] which states that for any given tensor $A_{ijk}...$, the following is true,

$$\int_V \frac{\partial}{\partial x_p} A_{ijk}...dV = \int_S A_{ijk}...n_p dS \tag{4.13}$$

equation (4.12) can be written as

$$\begin{aligned}
\frac{1}{2}\int_S t_i u_i dS &= \frac{1}{2}\int_V \frac{\partial}{\partial x_j}\left(\sigma_{ij}u_j\right) dV \\
&= \frac{1}{2}\int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j} dV \\
&= \frac{1}{2}\int_V \sigma_{ij} e_{ij} dV
\end{aligned} \tag{4.14}$$

In the above derivation, the equilibrium equations $\partial\sigma_{ij}/\partial x_j = 0$ have been invoked.

Comparing (4.14) with (4.11), we have

$$\underbrace{\frac{V}{2}C_{ij}\,\overline{\varepsilon}_i\overline{\varepsilon}_j}_{\text{equivalent homogeneous medium energy stored}} = \underbrace{\frac{1}{2}\int_V \sigma_i\varepsilon_i dV}_{\text{original body energy stored}} \tag{4.15}$$

$\sigma_1 = \sigma_{xx}$

From (4.15) we conclude that the equivalent homogeneous medium stores the same amount of elastic strain energy as in the original body. The effective moduli thus defined ensure the equivalence in strain energy between the equivalent medium and the original heterogeneous material.

In deriving the effective elastic constants or moduli, one must select the RVE and find the actual stress and strain distributions corresponding to the appropriate boundary conditions associated with uniform stress and strain in the equivalent homogeneous medium. This usually is not a simple task. Approximations are often employed in idealizing the geometry and the stress and strain fields in the RVE in order to simplify the mathematics involved. Voigt and Reuss models are two such examples.

4.2 VOIGT AND REUSS MODELS (Orthotropic laminate)

Consider a two phase composite for which the constituents are isotropic and are denoted by the f phase and m phase, respectively. Under an imposed macroscopically homogeneous strain $\bar{\varepsilon}_i$ on the representative volume element, the true strain field is not homogeneous. However, if we force

$$\varepsilon_i^f = \varepsilon_i^m = \bar{\varepsilon}_i \tag{4.16}$$

then from (4.5) we obtain

$$\begin{aligned} \bar{\sigma}_i &= \frac{1}{V}\left(\int_{V_f} C_{ij}^f\, \bar{\varepsilon}_j\, dV + \int_{V_m} C_{ij}^m\, \bar{\varepsilon}_j\, dV\right) \\ &= \left(c_f C_{ij}^f + c_m C_{ij}^m\right)\bar{\varepsilon}_j \end{aligned} \tag{4.17}$$

where $c_f = V_f/V$ is the fiber volume fraction, $c_m = V_m/V$ is the matrix volume fraction, and C_{ij}^f and C_{ij}^m are the elastic constants for the two phases, respectively. The effective elastic constants for the composite are obtained as

$$C_{ij}^V = c_f C_{ij}^f + c_m C_{ij}^m \tag{4.18}$$

which have the same form as that for the rule of mixtures. This uniform strain assumption was first proposed by Voigt [4.2] in connection with the related polycrystal problem.

The dual assumption is that the stress is uniform in the RVE, i.e.,

$$\sigma_i^f = \sigma_i^m = \bar{\sigma}_i \tag{4.19}$$

Using (4.6) and (4.19), the following relations are obtained

$$S_{ij}^R = c_f S_{ij}^f + c_m S_{ij}^m \tag{4.20}$$

$c_f S_{ij}^f + (1-c_f) S_{ij}^m$

The uniform stress assumption was originally proposed by Reuss [4.3] for application to a polycrystal. A good discussion on these two models was given by Hill [4.4].

In general, neither assumption is correct. The implied stresses according to the Voigt assumption of constant strain cannot satisfy the stress continuity condition at the phase boundaries, and the implied Reuss strains cannot satisfy the displacement continuity condition. It should be noted that the elastic constants C_{ij}^V and compliances S_{ij}^R given by (4.18) and (4.20), respectively, are not inverses of each other as they are estimated based on different deformation and stress fields in the composite.

Consider a representative volume element (RVE) of a macroscopically homogeneous and isotropic solid which is subjected to a state of hydrostatic pressure p. The average stresses are

$$\bar{\sigma}_{xx} = \bar{\sigma}_{yy} = \bar{\sigma}_{zz} = -p \tag{4.21}$$

$$\bar{\sigma}_{xy} = \bar{\sigma}_{xz} = \bar{\sigma}_{yz} = 0 \tag{4.22}$$

Using the stress-strain relations for isotropic solids, (2.81), we have

$$-3p = \bar{\sigma}_{xx} + \bar{\sigma}_{yy} + \bar{\sigma}_{zz} = (C_{11} + 2C_{12})\,\bar{\varepsilon}_0 \tag{4.23}$$

where

$$\bar{\varepsilon}_0 = \bar{\varepsilon}_{xx} + \bar{\varepsilon}_{yy} + \bar{\varepsilon}_{zz}$$

is the volume change per unit volume of the representative volume element. The bulk modulus K is defined as

$$K = -\frac{p}{\bar{\varepsilon}_0} = \frac{C_{11} + 2C_{12}}{3} \tag{4.24}$$

If the strain-stress relations of (4.8) are used, then

$$\bar{\epsilon}_0 = \bar{\epsilon}_{xx} + \bar{\epsilon}_{yy} + \bar{\epsilon}_{zz} = -3\,(S_{11} + 2S_{12})\,p \tag{4.25}$$

and the bulk modulus can be expressed in compliances as

$$K = \frac{1}{3\,(S_{11} + 2S_{12})} \tag{4.26}$$

Consider a particulate composite which is to be represented by an equivalent isotropic solid. Using the Voigt assumption, we obtain

$$C_{11}^V = c_f C_{11}^f + c_m C_{11}^m \quad , \quad C_{12}^V = c_f C_{12}^f + c_m C_{12}^m$$

and the effective bulk modulus which is obtained from (4.24) as

$$K^V = \frac{c_f}{3}\left(C_{11}^f + 2C_{12}^f\right) + \frac{c_m}{3}\left(C_{11}^m + 2C_{12}^m\right) = c_f K_f + c_m K_m \tag{4.27}$$

This is of the same form as that for C_{ij}^V in (4.18).

If the Reuss uniform stress is imposed throughout the RVE, then we obtain

$$S_{11}^R = c_f S_{11}^f + c_m S_{11}^m \quad , \quad S_{12}^R = c_f S_{12}^f + c_m S_{12}^m$$

Substitution of the above equations into (4.26) yields

$$\begin{aligned} K^R &= \frac{1}{3c_f\left(S_{11}^f + 2S_{12}^f\right) + 3c_m\left(S_{11}^m + 2S_{12}^m\right)} \\ K^R &= \frac{1}{\dfrac{c_f}{K_f} + \dfrac{c_m}{K_m}} \end{aligned} \tag{4.28}$$

A similar examination of simple shear leads to the Voigt and Reuss estimates of the effective shear modulus as

$$G^V = c_f G_f + c_m G_m \tag{4.29}$$

and

$$G^R = \frac{1}{\dfrac{c_f}{G_f} + \dfrac{c_m}{G_m}}$$

respectively.

The difference between the estimates can be put as

$$K^V - K^R = \frac{(K_f - K_m)^2}{\dfrac{K_f}{c_f} + \dfrac{K_m}{c_m}} > 0 \tag{4.30}$$

with a similar expression for the shear moduli. Thus, the Voigt values always exceed the Reuss ones. The difference becomes large if the rigidities between the two constituent phases differ substantially.

4.3 A HYBRID VOIGT-REUSS MODEL

For composites with simple regular internal geometries, one may be able to determine with a certain degree of accuracy whether a mode of deformation is better approximated by constant stress (Reuss model) or constant strain (Voigt model) or a combination of both. For the purpose of illustration, let us consider an idealized two dimensional fiber reinforced composite as shown in Fig. 4.1. The fibers are assumed to have a rectangular cross section, and the medium is assumed to be of unit thickness. A state of plane stress parallel to $x_1 - x_2$ plane is assumed. The fibers and matrix are assumed to be isotropic with the elastic constants E_f, G_f, ν_f and E_m, G_m, ν_m, respectively.

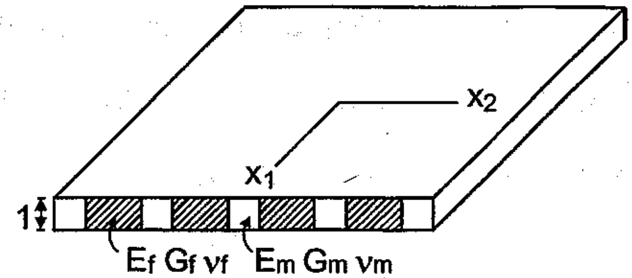


Figure 4.1 An idealized 2 D model for fiber composites

The composite is to be modeled as a 2 D orthotropic solid whose stress strain relations are given by

$$\left\{\begin{array}{c}\bar{\sigma}_{11}\\ \bar{\sigma}_{22}\\ \bar{\sigma}_{12}\end{array}\right\}=\left[\begin{array}{ccc}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{array}\right]\left\{\begin{array}{c}\bar{\varepsilon}_{11}\\ \bar{\varepsilon}_{22}\\ \bar{\gamma}_{33}\end{array}\right\} \tag{4.31}$$

or

$$\left\{\begin{array}{c}\bar{\varepsilon}_{11}\\ \bar{\varepsilon}_{22}\\ \bar{\gamma}_{12}\end{array}\right\}=\left[\begin{array}{ccc}\dfrac{1}{E_1} & -\dfrac{\nu_{12}}{E_1} & 0\\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0\\ 0 & 0 & \dfrac{1}{G_{12}}\end{array}\right]\left\{\begin{array}{c}\bar{\sigma}_{11}\\ \bar{\sigma}_{22}\\ \bar{\sigma}_{12}\end{array}\right\} \quad \text{along fiber} \tag{4.32}$$

Within the RVE, the following are assumed:

$$\begin{aligned}
\varepsilon_{11}^{f} &= \varepsilon_{11}^{m}=\bar{\varepsilon}_{11}: && \text{constant strain} \quad \text{(deformation is constant)} && (4.34a)\\
\sigma_{22}^{f} &= \sigma_{22}^{m}=\bar{\sigma}_{22}: && \text{constant stress} \quad \text{(stress is constant)} && (4.34b)\\
\sigma_{12}^{f} &= \sigma_{12}^{m}=\bar{\sigma}_{12}: && \text{constant stress} \quad \text{transversely} && (4.34c)
\end{aligned}$$

(Shear deformation is constant)

for any appropriate general deformation. Equation (4.34a) indicates non-separation between the fiber and the matrix, while (4.34b, c) satisfy the continuity of stresses at the interface of the fiber and the matrix. We arrive at these conditions often via our physical intuition.

Since the stress and strain are constant in each constituent, integrations in (4.3) and (4.4) can be carried out to yield the following relations.

$$\begin{aligned}
\bar{\sigma}_{11} &= c_f\sigma_{11}^{f}+c_m\sigma_{11}^{m} && (4.35a)\\
\bar{\varepsilon}_{22} &= c_f\varepsilon_{22}^{f}+c_m\varepsilon_{22}^{m} && (4.35b)\\
\bar{\gamma}_{12} &= c_f\gamma_{12}^{f}+c_m\gamma_{12}^{m} && (4.35c)
\end{aligned}$$

Equations (4.34) and (4.35) and the stress-strain relations of the fiber and matrix can be used to obtain the stress-strain relations for the equivalent orthotropic solid. Alternatively, the special deformation associated with each effective engineering modulus can be considered from which the particular modulus is obtained. The latter approach is taken in the following derivations.

- **Longitudinal Young's Modulus E_1 and Poisson's Ratio ν_{12}**

To determine the effective longitudinal modulus E_1 and the Poisson's ratio ν_{12} we consider a macroscopically simple tension, i.e.,

$$\bar{\sigma}_{11} \neq 0, \quad \bar{\sigma}_{22} = \bar{\sigma}_{12} = 0 \tag{4.36}$$

In view of (4.34) it is evident that both fiber and matrix are in simple tension. Thus,

$$\begin{aligned} \sigma_{11}^{f} &= E_f \varepsilon_{11}^{f} = E_f \, \bar{\varepsilon}_{11} && (4.37a) \\ \varepsilon_{22}^{f} &= -\nu_f \varepsilon_{11}^{f} = -\nu_f \, \bar{\varepsilon}_{11} && (4.37b) \\ \sigma_{11}^{m} &= E_m \varepsilon_{11}^{m} = E_m \, \bar{\varepsilon}_{11} && (4.38a) \\ \varepsilon_{22}^{m} &= -\nu_m \varepsilon_{11}^{m} = -\nu_m \, \bar{\varepsilon}_{11} && (4.38b) \end{aligned}$$

Substitution of (4.37a) and (4.38a) into (4.35a) yields

$$\bar{\sigma}_{11} = (c_f E_f + c_m E_m) \, \bar{\varepsilon}_{11}$$

Thus,

$$E_1 = c_f E_f + c_m E_m \tag{4.39}$$

Substitution of (4.37b) and (4.38b) into (4.35b) leads to

$$\bar{\varepsilon}_{22} = -\left(c_f \nu_f + c_m \nu_m\right) \bar{\varepsilon}_{11}$$

Hence,

$$\nu_{12} = c_f \nu_f + c_m \nu_m \tag{4.40}$$

- **In Plane Shear Modulus G_{12}**

Applying the macroscopic simple shear $\bar{\sigma}_{12} \neq 0$, $\bar{\sigma}_{11} = \bar{\sigma}_{22} = 0$, and using the constant stress condition given by (4.34c), we obtain from (4.35c)

$$\begin{aligned} \bar{\gamma}_{12} &= c_f \gamma_{12}^{f} + c_m \gamma_{12}^{m} \\ &= c_f \frac{\bar{\sigma}_{12}}{G_f} + c_m \frac{\bar{\sigma}_{12}}{G_m} \\ &= \left(\frac{c_f}{G_f} + \frac{c_m}{G_m} \right) \bar{\sigma}_{12} \end{aligned}$$

Thus, the in plane effective shear modulus is

$$\frac{1}{G_{12}} = \frac{c_f}{G_f} + \frac{c_m}{G_m} \tag{4.41}$$

- **Transverse Young's Modulus E_2**

Consider a state of macroscopic simple tension applied in the x_2 direction. We have

$$\bar{\sigma}_{22} \neq 0, \quad \bar{\sigma}_{11} = \bar{\sigma}_{12} = 0$$

In the fiber phase and matrix phase, we have

$$\varepsilon^f{}_{22} = \frac{1}{E_f}\,\bar{\sigma}_{22} - \frac{\nu_f}{E_f}\sigma^f_{11} \tag{4.42}$$

and

$$\varepsilon^m{}_{22} = \frac{1}{E_m}\,\bar{\sigma}_{22} - \frac{\nu_m}{E_m}\sigma^m_{11} \tag{4.43}$$

respectively, in which the constant stress assumption, (4.34b), has been invoked.

Substituting (4.42) and (4.43) into (4.35b), we obtain the average strain $\bar{\varepsilon}_{22}$as

$$\begin{aligned} \bar{\varepsilon}_{22} &= c_f\varepsilon^f_{22} + c_m\varepsilon^m_{22} \\ &= \left(\frac{c_f}{E_f} + \frac{c_m}{E_m}\right)\bar{\sigma}_{22} - \left(\frac{\nu_f}{E_f}c_f\sigma^f_{11} + \frac{\nu_m}{E_m}c_m\sigma^m_{11}\right) \end{aligned} \tag{4.44}$$

Since $\bar{\sigma}_{11} = 0$,(4.35a) leads to

$$c_m\,\sigma^m{}_{11} = -c_f\sigma^f_{11} \tag{4.45}$$

Upon substituting (4.45) into (4.44), we obtain

$$\bar{\varepsilon}_{22} = \left(\frac{c_f}{E_f} + \frac{c_m}{E_m}\right)\bar{\sigma}_{22} - \left(\frac{\nu_f}{E_f} - \frac{\nu_m}{E_m}\right)c_f\sigma^f_{11} \tag{4.46}$$

From the strain-stress relations of the fiber and the matrix, we have

$$\varepsilon^f_{11} = \frac{1}{E_f}\sigma^f_{11} - \frac{\nu_f}{E_f}\,\bar{\sigma}_{22} \tag{4.47}$$

$$\varepsilon^m_{11} = \frac{1}{E_m}\sigma^m_{11} - \frac{\nu_m}{E_m}\,\bar{\sigma}_{22} \tag{4.48}$$

Using the constant strain assumption $\varepsilon^f_{11} = \varepsilon^m_{11}$ in (4.48) together with (4.45), ε^f_{11} can be eliminated from (4.47) with the result

$$\sigma^f_{11} = \frac{c_m(\nu_f E_m - \nu_m E_f)}{c_f E_f + c_m E_m}\,\bar{\sigma}_{22} \tag{4.49}$$

Substitution of (4.49) into (4.46) yields the strain-stress relation for simple tension in the x_2-direction:

$$\bar{\varepsilon}_{22} = \left[\left(\frac{c_f}{E_f} + \frac{c_m}{E_m}\right) - \frac{c_f c_m \left(\nu_f E_m - \nu_m E_f\right)^2}{E_f E_m \left(c_f E_f + c_m E_m\right)}\right] \bar{\sigma}_{22}$$

Thus, the effective transverse Young's modulus E_2 is obtained as

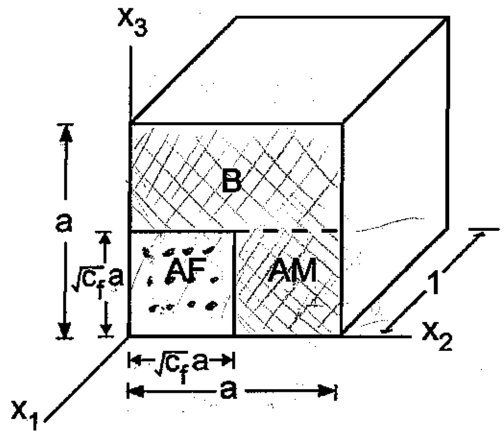
$$\frac{1}{E_2} = \left(\frac{c_f}{E_f} + \frac{c_m}{E_m}\right) - \frac{c_f c_m \left(\nu_f E_m - \nu_m E_f\right)^2}{E_f E_m \left(c_f E_f + c_m E_m\right)} \tag{4.50}$$

In general, $E_f > E_m$ and $c_f \approx c_m$. Consequently, (4.50) can be approximated by $\dfrac{1}{E_2} = \dfrac{c_f}{E_f} + \dfrac{c_m}{E_m}\left(1 - \nu_m^2\right)$.

Carbon fibers are transversely isotropic. The above procedure for evaluating engineering moduli can easily be extended to include anisotropy in the fiber and matrix.

4.4 A SQUARE FIBER MODEL

A more realistic RVE for unidirectional fiber composites is shown in Fig. 4.2. In this RVE, the round fiber is approximated by a square one. There are three regions, i.e., AF, AM and B as shown in the figure. Region AF is the fiber, regions AM and B are the matrix. To find the effective elastic moduli for the composite following the procedure described above, we first consider the effective moduli for composite region A which consists of AF and AM. This is precisely the model we have just investigated.



Q^A_{ij}

Q_f

Q_m

$Q_A = \sqrt{c_f} \cdot Q_f + (1-\sqrt{c_f})\, Q_m$

$Q_T = \sqrt{c_f} \cdot Q_A + (1-\sqrt{c_f})\, Q_B$

Figure 4.2 An improved RVE for fiber reinforced composites

The effective engineering moduli for the combined regions AF and AM can be obtained from (4.39), (4.40), (4.41), and (4.50). The corresponding reduced stiffnesses are denoted by Q^A_{ij} .

The next step is to consider the composite medium consisting of region B and region A with the latter already represented by an equivalent homogeneous medium with Q_{ij}^A. This can be accomplished by adopting the following assumptions regarding the three modes of deformation.

$$\varepsilon_{11}^A = \varepsilon_{11}^B = \bar{\varepsilon}_{11}: \quad \text{constant strain} \tag{4.51a}$$

$$\varepsilon_{22}^A = \varepsilon_{22}^B = \bar{\varepsilon}_{22}: \quad \text{constant strain} \tag{4.51b}$$

$$\gamma_{12}^A = \gamma_{12}^B = \bar{\gamma}_{12}: \quad \text{constant strain} \tag{4.51c}$$

The equivalent homogeneous medium in region A is a 2-D orthotropic solid whose stress strain relations assume the form of (4.32). Region B is a matrix region for which the stress strain relations are those for the isotropic matrix material.

In view of (4.51), it is evident that the present model in combining regions A and B is a Voigt model. Thus, the effective elastic constants for the composite are obtained as

$$Q_{ij} = c_A Q_{ij}^A + c_B Q_{ij}^B \tag{4.52}$$

where Q_{ij}^B are the plane stress reduced stiffnesses for the matrix, and c_A and c_B are the volume fractions of regions A and B, respectively. The relations between the plane stress reduced stiffnesses Q_{ij} and the elastic moduli are given by (3.2-3.3).

The effective elastic moduli for the composites are obtained from the effective Q_{ij} using the following relations

$$E_1 = \frac{Q_{11}Q_{22} - Q_{12}^2}{Q_{22}} \tag{4.53a}$$

$$E_2 = \frac{Q_{11}Q_{22} - Q_{12}^2}{Q_{11}} \tag{4.53b}$$

$$\nu_{12} = \frac{Q_{12}}{Q_{22}} \tag{4.53c}$$

$$G_{12} = Q_{66} \tag{4.53d}$$

References

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4.3 Reuss, A., "Berechnung der Fliessgrenze von Mischkristallen auf Grund der Plastizitatsbedingung für Einkristalle," *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 2, 1929, pp. 49-58.

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Problems

4.1. Derive the effective moduli E_1, E_2, G_{12},and ν_{12} for a unidirectionally fiber-reinforced composite with transversely isotropic fibers and an isotropic matrix. Use the hybrid Voigt-Reuss approach.

4.2. Consider the boron/aluminum with the following elastic properties.

Boron fiber	:	$E = 380\,\text{GPa}$,	$G = 172$ Gpa ,	$\nu = 0.1$
Aluminum	:	$E = 68\,\text{GPa}$,	$G = 26$ GPa ,	$\nu = 0.3$

Plot the effective moduli E_1, E_2, G_{12} and ν_{12} of the composite versus fiber volume fraction c_f using the following methods.

a) Voigt assumption

b) Reuss assumption

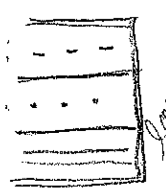
c) Hybrid model

d) Square fiber model

Chapter 5
ELASTIC ANALYSIS OF COMPOSITE LAMINATES

Unidirectionally reinforced fiber composites have superior properties only in the fiber direction. In practical applications, laminae with various fiber orientations are combined together to form laminated composites which are capable of carrying loads of multiple directions. Due to the lamination, the material properties of a laminate become heterogeneous over the thickness. Further, due to the arbitrary fiber orientations of the laminae, the laminate may not possess orthotropy as each constituent lamina does.

5.1 NOTATION FOR STACKING SEQUENCE



laminate

3 laminae also called plies.

A laminate consists of a number of laminae of different fiber orientations. A composite ply is the basic element in constructing a laminate. Each lamina may contain one or more plies of the same fiber orientation. The laminate properties depend on the lamina fiber orientation as well as its position in the laminate (the stacking sequence). To describe a laminate, the fiber orientation and position of each laminate must be accurately specified.

To achieve the above purpose, a global coordinate system, (x, y, z), must be established. Let the x-y plane be parallel to the plane of the laminate and the z-axis be in the thickness direction. The fiber orientation (θ) is measured relative to the x-axis as shown in Fig. 3.1. The positions of the plies are listed in sequence starting from one face of the laminate to the other face along the positive z-direction. An example is shown in Fig. 5.1 for [0/0/45/−45].

In practice, layup is not arbitrary; it often possesses certain repetitions and symmetry. To avoid lengthy expressions, abbreviated notations are used to specify the stacking sequence. The following are some abbreviated notations introduced to indicate ply or sublaminate repetition and symmetry in layup.

Symmetry: If the layup is symmetric with respect to the midplane of the laminate, then only half of the plies are specified, and the other half are included by a subscript "s" indicating symmetric layup. An example is $[0/90/+45/-45]_S$ or $[0/90/\pm 45]_S$ which stands for $[0/90/+45/-45/-45/+45/90/0]$.

Repetition: If a ply or sublaminate contiguously repeats itself n times in a laminate, then a subscript n is attached to the ply angle or the sublaminate group angles to indicate the repetitions. For example, $[0_2/90_2]$ stands for $[0/0/90/90]$, and

$[(0/90)_2/\pm 45_2]$ stands for [0/90/0/90/45/45/-45/−45].

Additional examples are given below.

$[0/\pm 45/\underline{0}]_S$ stands for [0/45/−45/0/−45/45/0] where an underline is used to indicate the ply right on the plane of symmetry.

$[(0/90)_2]_S$ or $[0/90]_{2S}$ stands for [0/90/0/90/90/0/90/0].

$[(0/90)_S]_2$ stands for [0/90/90/0/0/90/90/0].

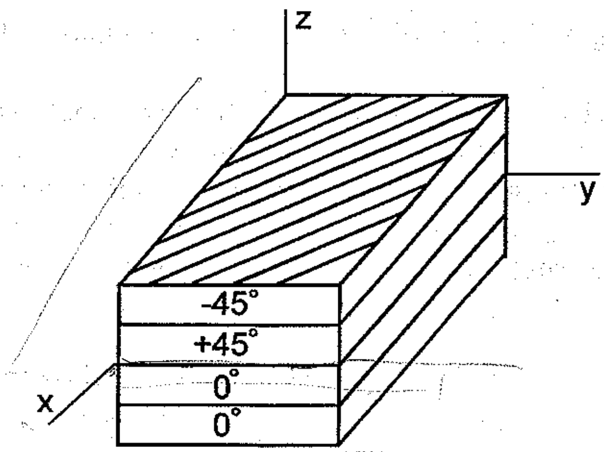


Figure 5.1 Coordinate system and stacking sequence

5.2 CLASSICAL LAMINATED PLATE THEORY

Consider a laminated plate consisting of a number of fiber reinforced laminae. The Cartesian coordinate system is set up as shown in Fig. 5.2. The x-y plane is located at the midplane of the plate. Let u, v, w be the displacement components in the x, y, and z directions, respectively. In general, u, v, w are functions of x, y, and z. Expand the displacement components in terms of power series of z, we have

why?

$$\begin{aligned} u(x,y,z) &= \sum_{i=0}^{\infty} z^i u_i(x,y) \\ v(x,y,z) &= \sum_{i=0}^{\infty} z^i v_i(x,y) \\ w(x,y,z) &= \sum_{i=0}^{\infty} z^i w_i(x,y) \end{aligned} \tag{5.1}$$

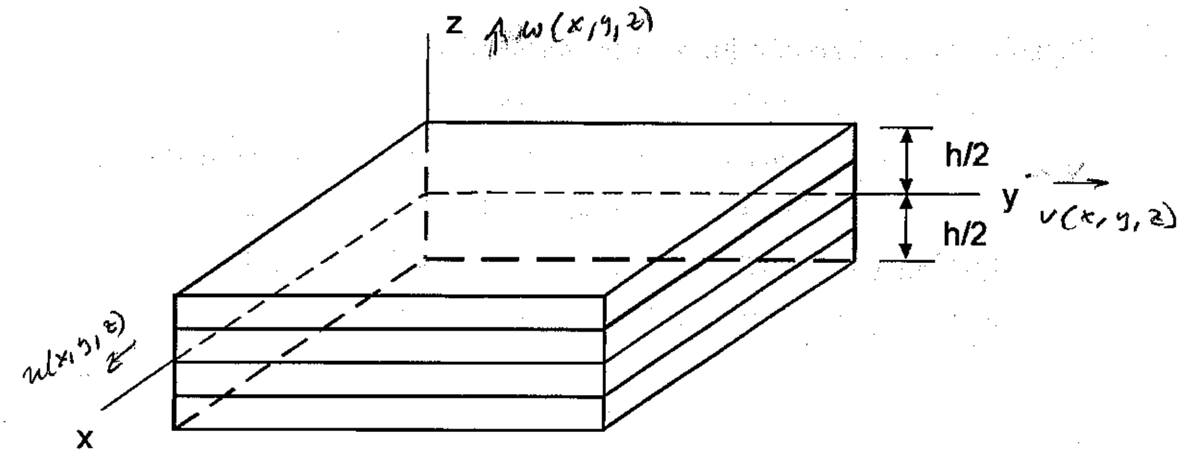


Figure 5.2 Coordinate system for laminated plate

plane stress

For laminated plates with thicknesses that are small as compared with the lateral dimensions, the variation of displacement over the thickness is small. Consequently, good approximation can be achieved by retaining the first few terms in the series expansion given by (5.1). In view of this, we consider the following approximate plate displacements (using ψ_x and ψ_y to replace u_1 and v_1, respectively):

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\psi_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\psi_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{5.2}$$

where u_0, v_0, and w_0 are identified as the displacement components of the midplane. Equations (5.2) also indicate that the transverse displacement w is assumed to be constant over the thickness. It should be noted that according to the displacement assumption (5.2), u and v are linear functions of z; this indicates that plane sections will remain plane after deformation.

The transverse shear strains γ_{xz} and γ_{yz} corresponding to the assumed plate displacements are

$$\begin{aligned} \gamma_{xz} &= \frac{\partial w_0}{\partial x} + \psi_x \\ \gamma_{yz} &= \frac{\partial w_0}{\partial y} + \psi_y \end{aligned} \tag{5.3}$$

If we further require that the deformed midplane remains perpendicular to the plane sections, i.e., the transverse shear strains vanish, then

$$\begin{aligned} \psi_x &= -\frac{\partial w_0}{\partial x} \\ \psi_y &= -\frac{\partial w_0}{\partial y} \end{aligned} \tag{5.4}$$

Substitution of (5.4) into (5.2) yields the displacements in classical plate theory:

EOM.

$$\boxed{\begin{aligned} u &= u_0 - z\frac{\partial w_0}{\partial x} \\ v &= v_0 - z\frac{\partial w_0}{\partial y} \\ w &= w_0 \end{aligned}} \tag{5.5}$$

The strains corresponding to these approximate displacements are

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \varepsilon_x^0 + z\kappa_x \\ \varepsilon_{yy} &= \varepsilon_y^0 + z\kappa_y \\ \gamma_{xy} &= \gamma_{xy}^0 + z\kappa_{xy} \end{aligned}} \tag{5.6}$$

where

$$\begin{aligned} \varepsilon_x^0 &= \frac{\partial u_0}{\partial x} \quad , \quad \varepsilon_y^0 = \frac{\partial v_0}{\partial y} \quad , \quad \gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\ \kappa_x &= -\frac{\partial^2 w_0}{\partial x^2} \quad , \quad \kappa_y = -\frac{\partial^2 w_0}{\partial y^2} \quad , \quad \kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y} \end{aligned} \tag{5.7}$$

Thus, the strains of the laminate are continuous over the thickness, and are described by the in-plane strains $\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0$ and the curvatures $\kappa_x, \kappa_y, \kappa_{xy}$ of the mid-surface.

5.3 PLATE CONSTITUTIVE EQUATIONS

Although the strains are continuous over the thickness of the laminate, the stresses in the laminae are, in general, discontinuous across the interfaces due to different material properties resulting from different fiber orientations. For the kth lamina, the stress components are given by

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\}_k = \left[\begin{array}{ccc} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{array} \right]_k \left[\left\{ \begin{array}{c} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{array} \right\} + z \left\{ \begin{array}{c} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right\} \right] \tag{5.8}$$

σ

It is apparent that analyzing each layer individually is a cumbersome task. Also, since the displacements in the layers are given by three "global" functions, u_0, v_0, w_0, it is desirable that some "plate force" quantities that do not identify with the individual laminae be employed. These quantities are usually called the plate resultant forces $\{N\}$ and moments $\{M\}$ defined by

$$\left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array} \right\} = \int_{-h/2}^{h/2} \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} dz \tag{5.9}$$

and

$$\left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = \int_{-h/2}^{h/2} \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} z dz \tag{5.10}$$

where h denotes the thickness of the plate, and the stress components σ_{xx}, σ_{yy} and σ_{xy} assume the values of $\sigma_{xx}^{(k)}, \sigma_{yy}^{(k)}$ and $\sigma_{xy}^{(k)}$ if z is located in the kth layer. The resultant forces and moments are depicted in Figs. 5.3 and 5.4.

Wed

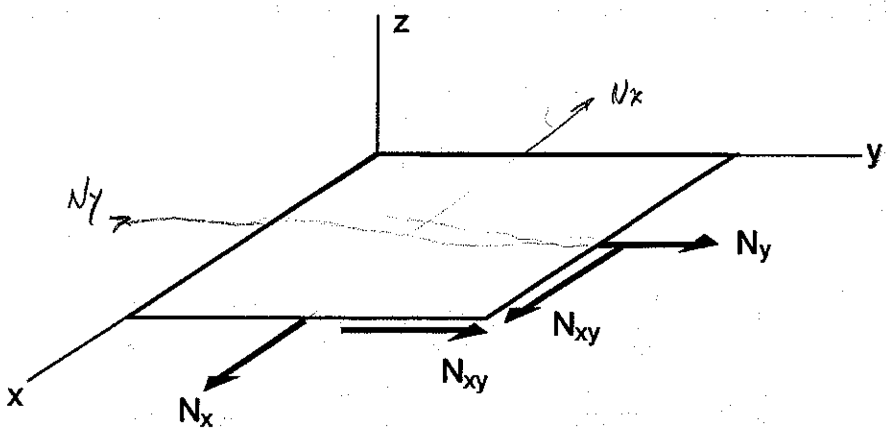


Figure 5.3 Resultant forces

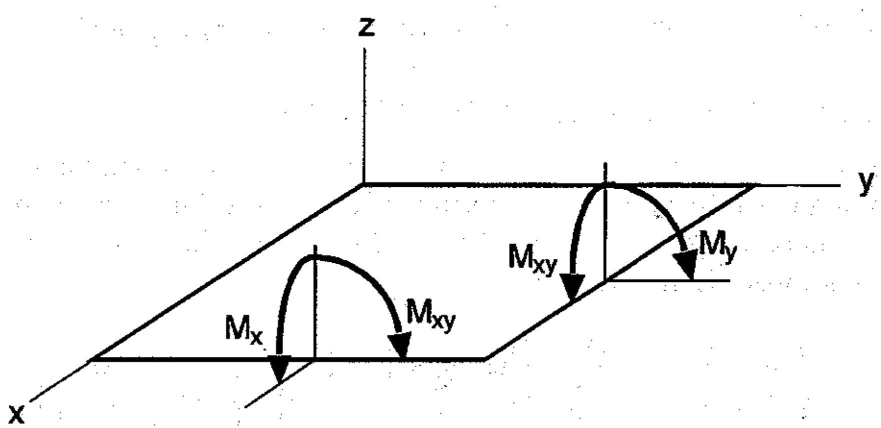


Figure 5.4 Bending and twisting moments

If the kth layer occupies the region from $z = z_{k-1}$ to $z = z_k$ (see Fig. 5.5), the integrals in (5.9) and (5.10) can be expressed as

inplane forces

$$\left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array} \right\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\}_k dz \tag{5.11}$$

and

in plane moments

$$\left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\}_k z dz \tag{5.12}$$

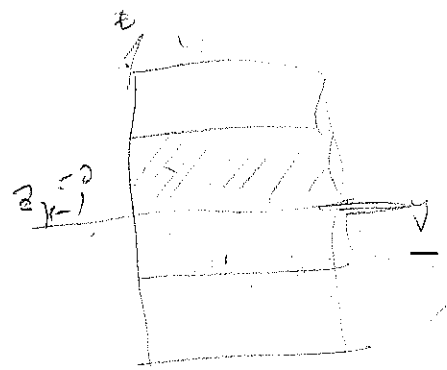
where n is the total number of layers in the laminate. Substituting (5.8) into (5.11) and (5.12), we obtain

$$\left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array} \right\} = \sum_{k=1}^{n} \left[\bar{Q}\right]_k \left(\int_{z_{k-1}}^{z_k} \left\{ \begin{array}{c} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{array} \right\} dz + \int_{z_{k-1}}^{z_k} \left\{ \begin{array}{c} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right\} z dz \right) \tag{5.13}$$

and

$$\left\{ \begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array} \right\} = \sum_{k=1}^{n} \left[\bar{Q}\right]_k \left(\int_{z_{k-1}}^{z_k} \left\{ \begin{array}{c} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{array} \right\} z dz + \int_{z_{k-1}}^{z_k} \left\{ \begin{array}{c} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right\} z^2 dz \right) \tag{5.14}$$

where n is the number of laminae in the laminate.



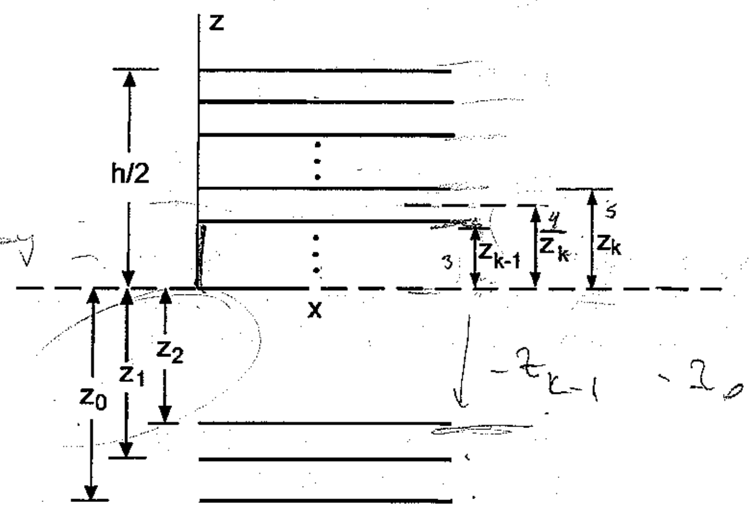


Figure 5.5 Lamina coordinates

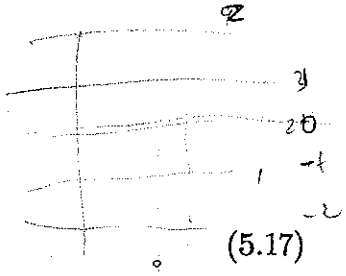
Note that the quantities $\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0, \kappa_x, \kappa_y$, and κ_{xy} are independent of z. Hence, the integrations in (5.13) and (5.14) can be performed. The results can be combined into the following form

$$\left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \\ -- \\ M_x \\ M_y \\ M_{xy} \end{array} \right\} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ -- \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right\} \tag{5.15}$$

where

$$(A_{ij},\ B_{ij},\ D_{ij}) = \int_{-h/2}^{h/2} \bar{Q}_{ij}^{(k)} \left(1, z, z^2\right) dz \tag{5.16}$$

or, explicitly,



$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} \left(z_k - z_{k-1}\right) \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} \left(z_k^2 - z_{k-1}^2\right) \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} \left(z_k^3 - z_{k-1}^3\right) \end{aligned} \tag{5.17}$$

$z_1 - z_0$
$z_2 - z_1$

Denoting the thickness and the distance to the centroid of the kth lamina by t_k and $\bar{z}_k$, respectively, (see Fig. 5.5) we can also write (5.17) as $z_3 - z_2$

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)}\, t_k \\ B_{ij} &= \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)}\, t_k\, \bar{z}_k \\ D_{ij} &= \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} \left(t_k \bar{z}_k^2 + \frac{t_k^3}{12}\right) \end{aligned} \tag{5.18}$$

$z_2 - z_1$
$z_3 - z_2$
$z_4 - z_3$

Coefficients A_{ij} are called extensional stiffnesses; B_{ij}, coupling stiffnesses; and D_{ij}, bending stiffnesses. These stiffnesses can also be written in terms of the invariants $U_1, U_2, \cdots, U_5$ by using relations (3.17). For A_{ij}, we have

$$\begin{aligned} A_{11} &= V_{0A}U_1 + V_{1A}U_2 + V_{3A}U_3 \\ A_{12} &= V_{0A}U_4 - V_{3A}U_3 \\ A_{16} &= \frac{1}{2}V_{2A}U_2 + V_{4A}U_3 \\ A_{22} &= V_{0A}U_1 - V_{1A}U_2 + V_{3A}U_3 \\ A_{26} &= \frac{1}{2}V_{2A}U_2 - V_{4A}U_3 \\ A_{66} &= V_{0A}U_5 - V_{3A}U_3 \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} V_{0A} &= h \\ V_{1A} &= \int_{-h/2}^{h/2} \cos 2\theta dz = \sum_{k=1}^{n} t_k \cos 2\theta_k \end{aligned}$$

$$V_{2A} = \int_{-h/2}^{h/2} \sin 2\theta dz = \sum_{k=1}^{n} t_k \sin 2\theta_k \tag{5.20}$$

$$V_{3A} = \int_{-h/2}^{h/2} \cos 4\theta dz = \sum_{k=1}^{n} t_k \cos 4\theta_k$$

$$V_{4A} = \int_{-h/2}^{h/2} \sin 4\theta dz = \sum_{k=1}^{n} t_k \sin 4\theta_k$$

Similar expressions for B_{ij} and D_{ij} are obtained from (5.19) by replacing V_{iA} (i = 0, 1, ...4) with V_{iB} and V_{iD}, respectively. The coefficients V_{iB} and V_{iD} are defined as

$$V_{[0B,1B,2B,3B,4B]} = \int_{-h/2}^{h/2} [1, \cos 2\theta, \sin 2\theta, \cos 4\theta, \sin 4\theta]\, z dz \tag{5.21}$$

$$V_{[0D,1D,2D,3D,4D]} = \int_{-h/2}^{h/2} [1, \cos 2\theta, \sin 2\theta, \cos 4\theta, \sin 4\theta]\, z^2 dz \tag{5.22}$$

Symbolically, (5.15) is usually expressed in the form

in-plane load
bending moment

$$\left\{ \begin{array}{c} N \\ -- \\ M \end{array} \right\} = \left[\begin{array}{ccc} A & | & B \\ -- & -- & -- \\ B & | & D \end{array} \right] \left\{ \begin{array}{c} \varepsilon^0 \\ -- \\ \kappa \end{array} \right\} \tag{5.23}$$

Consider the case where $[B] = [0]$, then (5.23) reduces to

$$\left\{ \begin{array}{c} \varepsilon^0 \\ \kappa \end{array} \right\} = \left[\begin{array}{cc} A & B \\ B & D \end{array} \right]^{-1} \left\{ \begin{array}{c} N \\ \overline{M} \end{array} \right\}$$

$$\{N\} = [A]\{\varepsilon^0\}$$
$$\{M\} = [D]\{\kappa\}$$

As a result, the in-plane forces $\{N\}$ and strains $\{\varepsilon^0\}$ are uncoupled from the bending moments $\{M\}$ and the out of plane deflection associated with the curvatures $\{\kappa\}$. In other words, when the plate is subjected only to in-plane forces, no out-of-plane deformations would occur. Similarly, when only bending moments are applied, no extensions of the mid-surface of the plate would be induced.

Obviously, if $[B]$ is not null, then coupling between extension and bending exists. Thus, stretching a laminated plate by in-plane forces may also produce bending and twisting in the plate and vice versa.

5.4 SPECIAL CLASSES OF LAMINATES

Symmetric Laminates $[B] = 0$ from definition of B matrix

decoupled: separated $\begin{cases} \{N\} = [A]\{\varepsilon^0\} \\ \{M\} = [D]\{\kappa\} \end{cases}$ inplane forces produce inplane deformation

In-plane isotropy: [A] is invariant with respect to coordinate transformation.

Fully isotropic laminate: [A] & [D] are invariant.

In a symmetric laminate both the geometry and material are symmetric about the midplane. Due to these symmetries, it is not difficult to see from (5.18) that the coupling stiffnesses B_{ij} vanish. The uncoupling of extension and bending as a result of $B_{ij} = 0$ is often desired because the laminate is easier to analyze and is free from distortion resulting from residual stresses (curing stresses).

N_x, σ_x N_y, σ_y

A special case is the single-layered plate which is basically a plate of an orthotropic material. In addition to $B_{ij} = 0$, we also have

$\bar{\sigma}_x = \frac{N}{t}$, $\bar{\sigma}_y = \frac{N_y}{t}$

$\bar{\tau}_{xy} = \frac{N_{xy}}{t}$

$$
\begin{aligned}
A_{ij} &= \bar{Q}_{ij}\, h \\
D_{ij} &= \frac{1}{12}\, \bar{Q}_{ij}\, h^3
\end{aligned}
\tag{5.24}
$$

If the material is isotropic, then the above expressions further simplify to

$$
\begin{aligned}
A_{11} &= A_{22} = \frac{Eh}{1-\nu^2} \quad , \quad A_{12} = \frac{\nu Eh}{1-\nu^2} \\
A_{66} &= Gh \quad , \quad A_{16} = A_{26} = 0 \\
D_{11} &= D_{22} = \frac{Eh^3}{12\,(1-\nu^2)} \quad , \quad D_{12} = \nu D_{11} \\
D_{66} &= \frac{1}{12} G h^3 \quad , \quad D_{16} = D_{26} = 0
\end{aligned}
\tag{5.25}
$$

Effective Moduli for Symmetric Laminates

A symmetric laminate under in-plane loading can be treated as an equivalent homogeneous anisotropic solid by introducing the average stresses

$$
\boxed{\bar{\sigma}_x = N_x/h \quad , \quad \bar{\sigma}_y = N_y/h \quad , \quad \bar{\sigma}_{xy} = N_{xy}/h} \quad \text{average stress}
\tag{5.26}
$$

The plate constitutive equation for in-plane loading can be written as

$$
\frac{1}{h}\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \frac{1}{h} \Rightarrow
\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{xy} \end{Bmatrix} = \frac{1}{h}[A] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}
\tag{5.27}
$$

stress strain relation for the whole body.

also: $\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = h[A']\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{xy} \end{Bmatrix}$

Equation (5.27) indicates that the laminate is effectively a 2-D anisotropic solid in-plane stress and $[A]/h$ is the effective elastic constant matrix. The inverse relation of (5.27) is

$$
\boxed{\left\{\varepsilon^0\right\} = h\,[A']\,\{\bar{\sigma}\}} \quad \text{strain - stress}
\tag{5.28}
$$

where

$$
\boxed{[A'] = [A]^{-1}}
$$

The components A'_{ij} are given by

$$
A'_{11} = \left(A_{22}A_{66} - A_{26}^2\right)/\Delta
$$

In term of engineering moduli

$$
\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix} = \begin{bmatrix} 1/E_x & -\nu_{xy}/E_y & \eta_{xy,x}/G_{xy} \\ -\nu_{xy}/E_x & 1/E_y & \eta_{xy,y}/G_{xy} \\ \eta_{x,xy}/E_x & \eta_{x,xy}/E_y & 1/G_{xy} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{xy} \end{Bmatrix} \rightarrow \underline{5.30}
$$

$$
\begin{aligned}
A'_{12} &= (A_{16}A_{26} - A_{12}A_{66})/\Delta \\
A'_{22} &= \left(A_{11}A_{66} - A_{16}^2\right)/\Delta \\
A'_{16} &= (A_{12}A_{26} - A_{22}A_{16})/\Delta \\
A'_{26} &= (A_{12}A_{16} - A_{11}A_{26})/\Delta \\
A'_{66} &= \left(A_{11}A_{22} - A_{12}^2\right)/\Delta
\end{aligned}
\tag{5.29}
$$

where $A'_{11} = (A_{22}A_{66} - A_{26}^2)/\Delta$

$$\Delta = |A_{ij}|$$

Comparing (5.28) with (3.14), we can relate the components A'_{ij} to the effective engineering moduli for the laminate as

general symmetric laminates

$$
\begin{aligned}
E_x &= \frac{1}{hA'_{11}} \quad , \quad E_y = \frac{1}{hA'_{22}} \\
\nu_{xy} &= -\frac{A'_{12}}{A'_{11}} \quad , \quad \nu_{yx} = -\frac{A'_{12}}{A'_{22}} \\
\eta_{xy,x} &= \frac{A'_{16}}{A'_{66}} \quad , \quad \eta_{xy,y} = \frac{A'_{26}}{A'_{66}} \\
G_{xy} &= \frac{1}{hA'_{66}}
\end{aligned}
\tag{5.30}
$$

If a symmetric laminate also possesses the property $A_{16} = A_{26} = 0$ (e g , $[0/90]_s$ and $[\pm 45]_s$), then the effective moduli can be explicitly expressed as

for $[0/90]_s$ $[\pm 45]_s$ or similar only

$$
\begin{aligned}
E_x &= (A_{11}A_{22} - A_{12}^2)/hA_{22} \\
E_y &= (A_{11}A_{22} - A_{12}^2)/hA_{11} \\
\nu_{xy} &= A_{12}/A_{22} \\
\nu_{yx} &= A_{12}/A_{11} \\
G_{xy} &= A_{66}/h
\end{aligned}
\tag{5.31}
$$

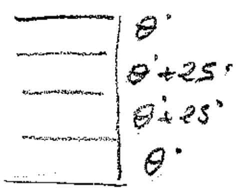
Symmetric laminates average moduli

Example 5.1 *Negative Poisson's Ratio in Laminates*

When a symmetric laminate is treated as a two-dimensional homogeneous solid in plane stress, it may exhibit some unusual properties that are not observed in other homogeneous solids. One of these is negative Poisson's ratios.

Consider the symmetric but unbalanced laminates $[\theta, \theta + 25°\,]_S$ where $\theta = 0°$ to 180°. The ply properties are given as

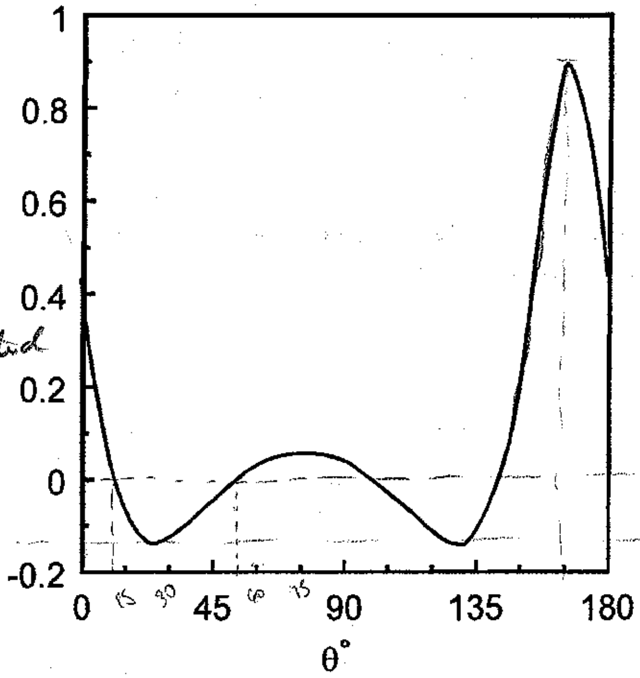
$$
\begin{aligned}
E_1 &= 180\,GPa \quad , \quad E_2 = 10\,GPa \quad , \quad G_{12} = 7\ GPa \\
\nu_{12} &= 0.28 \quad , \quad \text{Ply thickness} = 0.13\text{ mm}
\end{aligned}
$$



The apparent Poisson's ratio ν_{xy} can be calculated using (5.30). Figure 5.6 shows ν_{xy} as a function of θ. It is seen that negative values of ν_{xy} are possible. Also note that unusually high positive values of ν_{xy} can be produced.

Poisson's ratio.

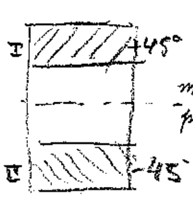
If the laminate is treated as a 2D layer, it can exhibit weird behavior.



negative Poisson's ratio.

Figure 5.6 Apparent Poisson's ratio ν_{xy} in $[\theta/\theta+25^o]_S$ laminates

Antisymmetric Laminates (Don't want because B might must be constant)



Nonsymmetric laminates may be used for fabricating pre–twisted components such as turbine blades. The material properties of an antisymmetric laminate is antisymmetric about the midplane. For a laminate consisting of similar fibrous plies to be antisymmetric, the number of plies must be even. For example, [+45/+30/-30/-45] is an antisymmetric laminate. It is then easy to see that a $+\theta^\circ$ layer always accompanies a $-\theta^\circ$ layer located at the symmetric position with respect to the midplane. From (3.10), we have

if $\bar{Q}_{16_I} = a$, $\bar{Q}_{16_{II}} = -a$ } same value but opposite sign because of antisymmetry

$$\left(\bar{Q}_{16}\right)_{+\theta} = -\left(\bar{Q}_{16}\right)_{-\theta}$$
$$\left(\bar{Q}_{26}\right)_{+\theta} = -\left(\bar{Q}_{26}\right)_{-\theta}$$

$B \neq 0$ 90° when cooled

$A_{16} = \sum Q_{16} = 0$

which lead to

$$A_{16} = A_{26} = D_{16} = D_{26} = 0 \qquad [B] \neq 0$$

Thus, neither extension-shear coupling nor bending-twisting coupling is present in antisymmetric laminates.

Cross-ply Laminates

A cross-ply laminate consists of an arbitrary number of plies each with fiber orientation of either 0° or 90° to the $x-$ axis. Since $\bar{Q}_{16}=\bar{Q}_{26}=0$ for both plies, thus, $A_{16}=A_{26}=B_{16}=B_{26}=D_{16}=D_{26}=0$. An antisymmetric cross ply laminate is one that has a 0° ply for every 90° ply at the symmetric position and vice versa. For such laminates, we have the additional properties

$$B_{22}=-B_{11} \quad , \quad B_{12}=B_{66}=0 \tag{5.32}$$

Balanced Laminates

If for every $+\theta$ ply there is a $-\theta$ ply in a laminate, then the laminate is called a balanced laminate. For laminates of this type, $A_{16}=A_{26}=0$, but, in general, $D_{16} \neq 0$ and $D_{26} \neq 0$.

[A] is invariant but not [D] matrix

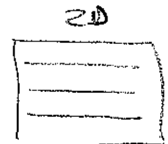
5.5 QUASI-ISOTROPIC LAMINATES (Special case of Symmetrical laminates)

How to make an isotropic laminate

For symmetric laminates, the in-plane deformation $\{\varepsilon^0\}$ and bending curvatures $\{\kappa\}$ are uncoupled. The in-plane load-deformation relation is given by $[A]$ as

$$\{N\}=[A]\left\{\varepsilon^0\right\} \tag{5.33}$$

2D



If $[A]$ is invariant with respect to coordinate transformation, then the laminate would have the same longitudinal stiffness in every direction; i.e., the in-plane laminate stiffness is isotropic. However, a laminate possessing an isotropic $[A]$ does not necessarily imply an isotropic $[D]$, thus the name quasi-isotropic laminate.

Use $\bar{Q}_{ij}$ matrix in terms of V_1, V_2 (3.17, 3.18)

$E_1, E_2, G_{12}, \nu_{12}$

Consider a symmetric laminate having N fiber orientations each of which consists of an equal number of composite plies. The total thickness of plies with the same fiber orientation is

or isotropic material. E, ν or G.

$$t=h/N$$

Using the expressions of $\bar{Q}_{ij}$ given in (3.~~18~~ 17), we have

A laminate becomes isotropic if $[A]_{ij}$ is invariant with respect to coordinate changes.

$$
\begin{aligned}
A_{11} &= \frac{h}{N}\left\{NU_1+U_2\sum_{k=1}^{N}\cos 2\theta_k+U_3\sum_{k=1}^{N}\cos 4\theta_k\right\} \\
A_{12} &= \frac{h}{N}\left\{NU_4-U_3\sum_{k=1}^{N}\cos 4\theta_k\right\} \\
A_{22} &= \frac{h}{N}\left\{NU_1-U_2\sum_{k=1}^{N}\cos 2\theta_k+U_3\sum_{k=1}^{N}\cos 4\theta_k\right\} \\
A_{16} &= \frac{h}{N}\left\{\frac{1}{2}U_2\sum_{k=1}^{N}\sin 2\theta_k+U_3\sum_{k=1}^{N}\sin 4\theta_k\right\} \\
A_{26} &= \frac{h}{N}\left\{\frac{1}{2}U_2\sum_{k=1}^{N}\sin 2\theta_k-U_3\sum_{k=1}^{N}\sin 4\theta_k\right\} \\
A_{66} &= \frac{h}{N}\left\{NU_5-U_3\sum_{k=1}^{N}\cos 4\theta_k\right\}
\end{aligned} \tag{5.34}
$$

$$= \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{(A_{11}-A_{12})}{2} \end{bmatrix}$$

Can an arrangement of fibers make the laminate behave as an isotropic material?

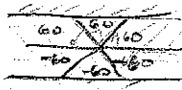
Laminates $\{N\}=[A]\{\varepsilon\}$

$\{\bar{\sigma}\}=\frac{1}{h}[A]\{\varepsilon\}$

then, what kind of A matrix guarantees the behavior of structure as isotropic mat?

to satisfy condition at least three angles are needed $N \geq 3$

if $N=3$, $\Delta\theta=\frac{\pi}{3}$



to make A_{ij} independent of coordinate system, all summations $= 0$ (Σ)

It is evident from (5.34) that the desired invariant property of $[A]$ can be achieved if

$$\begin{aligned} V_1 &= \sum_{k=1}^{N} \cos 2\theta_k = 0 \\ V_2 &= \sum_{k=1}^{N} \sin 2\theta_k = 0 \\ V_3 &= \sum_{k=1}^{N} \cos 4\theta_k = 0 \\ V_4 &= \sum_{k=1}^{N} \sin 4\theta_k = 0 \end{aligned} \tag{5.35}$$

simultaneously. To find conditions that satisfy (5.35) we consider the special situation where

$$\theta_k = (k-1)\,\pi/N \qquad k = 1, 2, \cdots, N \tag{5.36}$$

That is, the fiber orientations have an equal increment of π/N.

By using the relation for a complex function,

$$e^{inx} = \cos nx + i \sin nx \tag{5.37}$$

we can write

$$\begin{aligned} V_1 + iV_2 &= \sum_{k=1}^{N} e^{2(k-1)\pi i/N} \\ V_3 + iV_4 &= \sum_{k=1}^{N} e^{4(k-1)\pi i/N} \end{aligned} \tag{5.38}$$

Using the identity of geometric series

$$1 + e^{i\theta} + e^{2i\theta} + ... + e^{Ni\theta} = \frac{1 - e^{(N+1)\theta}}{1 - e^{i\theta}}$$

the above two geometric series are rewritten as

$$\begin{aligned} V_1 + iV_2 &= \frac{1 - e^{2\pi i}}{1 - e^{2\pi i/N}} \\ V_3 + iV_4 &= \frac{1 - e^{4\pi i}}{1 - e^{4\pi i/N}} \end{aligned} \tag{5.39}$$

Using the expressions, it is easy to verify $V_1 = V_3 = 1,\ V_2 = V_4 = 0$, for $N = 1$ and $V_1 = V_2 = V_4 = 0,\ V_3 = 2$ for $N = 2$. For $N \geq 3$, conditions (5.35) are satisfied simultaneously, and we have

$$\begin{aligned} A_{11} &= hU_1 \\ A_{12} &= hU_4 \\ A_{22} &= hU_1 \\ A_{16} &= A_{26} = 0 \\ A_{66} &= hU_5 = \frac{1}{2}(A_{11} - A_{12}) \end{aligned} \tag{5.40}$$

which indicate that the stiffness matrix $[A]$ is invariant with respect to coordinate transformation.

Apparently, the minimum number of fiber orientations to form a laminate with isotropic in-plane stiffness properties is $N = 3$. An example is the $[0/\pm 60]_s$ laminate which is called the $\pi/3$ laminate. The laminate $[\pm 45/0/90]_s$ (the $\pi/4$ laminate) belongs to the group of $N = 4$. Other quasi-isotropic laminates corresponding to higher values of N can be constructed in the same manner.

The in-plane stiffness of a quasi-isotropic laminate is similar to that of an isotropic solid, i.e.,

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & (A_{11} - A_{12})/2 \end{bmatrix} \tag{5.41}$$

5.6 LAMINAR STRESSES (Need to find stress to avoid failure)

The plate resultant forces and moments provide a convenient way to formulate the global governing equations for thin laminates. However, for prediction of the laminate strength, stresses in each lamina must be recovered.

If loads in terms of $\{N\}$ and $\{M\}$ are given, then the midplane deformation is obtained from

$$\boxed{\left\{ \begin{array}{c} \varepsilon^0 \\ -- \\ \kappa \end{array} \right\} = \left[\begin{array}{ccc} A & \mid & B \\ -- & -- & -- \\ B & \mid & D \end{array} \right]^{-1} \left\{ \begin{array}{c} N \\ -- \\ M \end{array} \right\}} \tag{5.42}$$

The stresses $\sigma_{xx}^{(k)}, \sigma_{yy}^{(k)}$, and $\sigma_{xy}^{(k)}$, in the kth lamina are calculated according to (5.8), i.e.,

$$\{\sigma\}_k = [\bar{Q}]_k(\{\varepsilon\}_k) = \boxed{\{\sigma\}_k = \left[\bar{Q}\right]_k \left(\left\{\varepsilon^0\right\} + z\left\{\kappa\right\}\right)} \tag{5.43}$$

If the plate displacements u_0, v_0, and w_0 are known, then $\{\varepsilon^0\}$ and $\{\kappa\}$ can be calculated from their relations with the plate displacements. The laminar stresses are then obtained from (5.43).

Example 5.2 *Laminar Stresses in a Quasi Isotropic Symmetric Laminate* $[\pm 45/0/90]_S$ *Subjected to a Uniaxial Load* N_x

Symmetric laminate under inplane loading.

The composite ply properties are

$\{N\} = [A]\{\varepsilon\}$

here $\{\varepsilon\}_k = \{\varepsilon^0\}_k$ b/c of no bending

$$
\begin{aligned}
E_1 &= 20 \times 10^6 \text{ psi} \quad , \quad E_2 = 1.4 \times 10^6 \text{ psi} \\
G_{12} &= 0.8 \times 10^6 \text{ psi} \quad , \quad \nu_{12} = 0.3 \\
t &= \text{ply thickness} = 0.005 \text{ inch}
\end{aligned}
\tag{a}
$$

The following properties are readily calculated.

$$
\left[\bar{Q}\right]_{0^\circ} = \begin{bmatrix} 20.13 & 0.42 & 0 \\ 0.42 & 1.41 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \times 10^6 \text{ psi} \tag{b}
$$

$$
\left[\bar{Q}\right]_{90^\circ} = \begin{bmatrix} 1.41 & 0.42 & 0 \\ 0.42 & 20.13 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \times 10^6 \text{ psi} \tag{c}
$$

$$
\left[\bar{Q}\right]_{\pm 45^\circ} = \begin{bmatrix} 6.4 & 4.8 & \pm 4.7 \\ 4.8 & 6.4 & \pm 4.7 \\ \pm 4.7 & \pm 4.7 & 5.2 \end{bmatrix} \times 10^6 \text{ psi} \tag{d}
$$

$$
[A] = \begin{bmatrix} 0.343 & 0.104 & 0 \\ 0.104 & 0.3453 & 0 \\ 0 & 0 & 0.119 \end{bmatrix} \times 10^6 \tag{e}
$$

The strains in the midplane are

$$
\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3.21 \\ -0.976 \\ 0 \end{Bmatrix} \times 10^{-6} N_x
$$

The laminar stresses are

$$
\begin{aligned}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_{0^\circ} &= \left[\bar{Q}\right]_{0^\circ} \left\{\varepsilon^0\right\} = \begin{Bmatrix} 64.19 \\ -0.02 \\ 0 \end{Bmatrix} N_x \\
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_{90^\circ} &= \left[\bar{Q}\right]_{90^\circ} \left\{\varepsilon^0\right\} = \begin{Bmatrix} 4.11 \\ -18.29 \\ 0 \end{Bmatrix} N_x \\
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_{\pm 45^\circ} &= \left[\bar{Q}\right]_{\pm 45^\circ} \left\{\varepsilon^0\right\} = \begin{Bmatrix} 15.85 \\ 9.15 \\ \pm 10.45 \end{Bmatrix} N_x
\end{aligned}
$$

The distributions of the laminar normal stresses are shown in Fig. 5.7. It is evident that the stress distributions from lamina to lamina are not continuous. Also note the large compressive stress σ_{yy} developed in the 90° lamina due to its inability to contract in the y-direction.

distribution of stresses are not uniform.

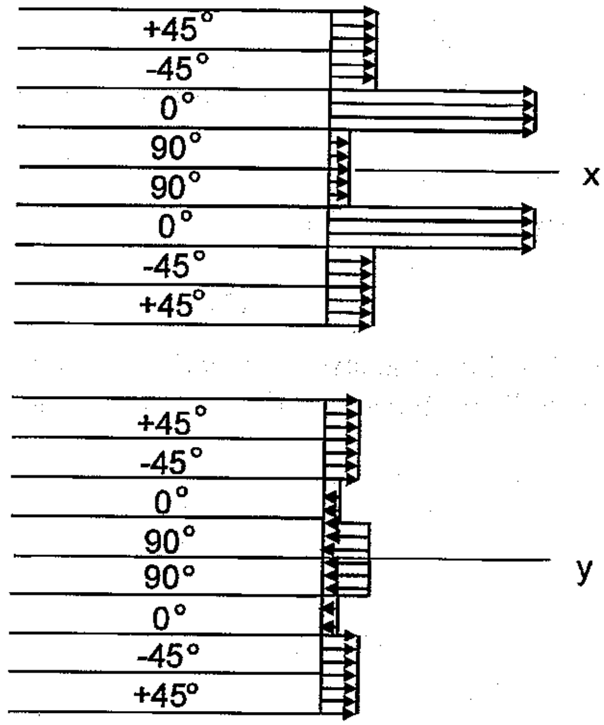
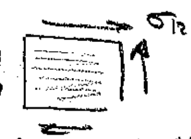


Figure 5.7 Normal stress distribution in the laminas



5.7 [±45°] LAMINATE (Enhance of shear testing)

The ±45° type of laminate is often used to provide greater shear rigidities of composite structures. For example, consider the $[\pm 45]_s$ laminate with the composite properties given in Example 5.2. We have

measure γ

measure of axis strain.

the $\sigma_{12} = \gamma \cdot G_{12}$

σ_{xx} ← σ_{xx}

$$[A] = \begin{bmatrix} 1.28 & 0.96 & 0 \\ 0.96 & 1.28 & 0 \\ 0 & 0 & 1.03 \end{bmatrix} \times 10^5 \ \text{lb/in}$$

$E_x = (\sigma_{12}..)$ (5.44)

measure. need to know E_1 E_2 ν_{12}

By treating this laminate as an equivalent homogeneous orthotropic solid in plane stress, the equivalent elastic moduli can be obtained from (5.31). We have

$$\begin{aligned} E_x = E_y = 2.8 \times 10^6 psi \quad &, \quad G_{xy} = 5.2 \times 10^6 psi \\ \nu_{xy} = \nu_{yx} = 0.75 \quad &, \quad \eta_{x,xy} = \eta_{xy,x} = 0 \end{aligned} \tag{5.45}$$

Comparing these moduli with those of the unidirectional composite, we note a significant increase in the shear rigidity. However, this is achieved at the expense of the longitudinal modulus E_x.

Determination of G_{12} Using ±45° Laminates

Consider the $[\pm 45]_s$ laminate subjected to a uniform uniaxial stress $\overline{\sigma}_x (= N_x/h) = \overline{\sigma}_0$ and $\overline{\sigma}_y = \overline{\sigma}_{xy} = 0$. It is not difficult to see that in both +45° and

$E_1 = 20\times10^6$ psi $G_{12} = 0.8 \times 10^6$ psi

$E_2 = 1.4 \times 10^8$ psi

$-45°$ laminae,

$$\sigma_{xx} = \sigma_0, \sigma_{yy} = 0, \sigma_{xy} \neq 0 \tag{5.46}$$

and that $(\sigma_{xy})_{-45°} = -(\sigma_{xy})_{+45°}$. From the coordinate transformation for stress, we have

$$\sigma_{12} = -\sin\theta\ \cos\theta\sigma_{xx} + \sin\theta\ \cos\theta\sigma_{yy} + (\cos^2\theta - \sin^2\theta)\ \sigma_{xy} \tag{5.47}$$

For $\theta = -45°$, the above equation yields

$$\boxed{\sigma_{12} = \frac{1}{2}\sigma_0} \tag{5.48}$$

Under uniaxial tension, the strains in both $+45°$ and $-45°$ are identical and $\gamma_{xy} = 0$. From the coordinate transformation law on strains,

6

$$\left\{\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{array}\right\} = [T_\epsilon] \left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array}\right\} \tag{5.49}$$

we obtain

$$\gamma_{12} = -2\sin\theta\cos\theta(\varepsilon_{xx} - \varepsilon_{yy}) \tag{5.50}$$

In the $-45°$ lamina, (5.50) gives

$$\boxed{\gamma_{12} = \varepsilon_{xx} - \varepsilon_{yy}} \tag{5.51}$$

laminate in x-y direction can be measured from tension test.

Thus,

$$G_{12} = \frac{\sigma_{12}}{\gamma_{12}} = \frac{\sigma_{12}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{5.52}$$

In view of (5.48) we conclude that

for the laminate $[\pm 45°]$ $E_x = \frac{\sigma_0}{\varepsilon_{xx}} \Rightarrow$

$$\boxed{G_{12} = \frac{\sigma_0}{2(\varepsilon_{xx} - \varepsilon_{yy})} = \frac{E_x}{2(1 + \nu_{xy})}} \tag{5.53}$$

$\frac{\sigma_0}{2\varepsilon_{xx}(1 - \frac{\varepsilon_{yy}}{\varepsilon_{xx}})}$

where $\nu_{xy} = -\varepsilon_{yy}/\varepsilon_{xx}$ is the effective Poisson's ratio of the laminate.

The relation (5.53) can be used to determine G_{12} from the tension test of a $[\pm 45]_S$ laminate specimen [5.1-5.2]. *for a $\pm 45°$ laminate, the Young's modulus & Poisson's ratio can be used during simple tension to find G_{12}

5.8 THERMAL STRESSES

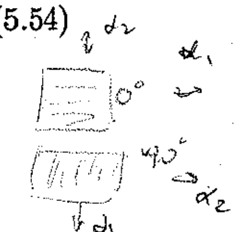
In unidirectional fiber composites, the thermal expansion is orthotropic. We denote the linear thermal expansion coefficients (strain/°C)in the fiber and transverse directions by α_1 and α_2, respectively. The thermal strains induced by a temperature change ΔT are

*Polymer composites are cured at high temperature (350°F)

$$\{\varepsilon^T\} = \left\{\begin{array}{c} \varepsilon_{11}^T \\ \\ \varepsilon_{22}^T \\ \\ \gamma_{12}^T \end{array}\right\} = \left\{\begin{array}{c} \alpha_1 \\ \alpha_2 \\ 0 \end{array}\right\} \Delta T \tag{5.54}$$

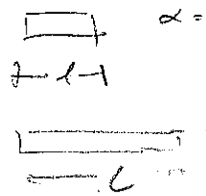
Laminate: suppose $[0/90]_s$

cooling down.



usually $\alpha_2 >> \alpha_1$

$\Delta L \; \{ \varepsilon \} \{ L$



$\alpha = \varepsilon / {}^{\circ}e$ or $\varepsilon / {}^{\circ}F$

$\Delta T = T - T_0$

reference temperature. $= \alpha \cdot \Delta T$

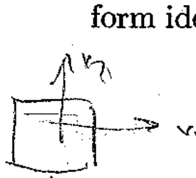
The mechanical strains that produce stresses are the total strains less the thermal strains. The stress strain relations for a plane stress unidirectional fiber composite are modified to

$$\{\sigma\} = [Q](\{\varepsilon\} - \{\varepsilon^T\}) \tag{5.55}$$

in reference to the material principal axes.

In view of (5.54), the vector of thermal expansion coefficients (α) should transform identically as the strains. That is, → fiber direction

2D



$\Delta T = 1$

$\varepsilon_{11}^T = \alpha_1$

$\varepsilon_{22}^T = \alpha_2$

$\gamma_{12}^T = 0$

$L = (\alpha \Delta T) \cdot L$

$$\boxed{\left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ 0 \end{array} \right\} = [T_\epsilon] \left\{ \begin{array}{c} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{array} \right\}} \tag{5.56}$$

* for fiber composites, we have

$\{\alpha\} = \left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{array} \right\}$

The inverse relation of (5.56) can readily be obtained as

3 planes of symmetry.

$$\begin{aligned} \alpha_x &= \alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta \\ \alpha_y &= \alpha_1 \sin^2\theta + \alpha_2 \cos^2\theta \\ \alpha_{xy} &= 2(\alpha_1 - \alpha_2)\cos\theta \sin\theta \end{aligned} \tag{5.57}$$

Then the transformed thermal strains in the $x-y$ coordinate system are

$$\left\{\varepsilon^T\right\} = \left\{ \begin{array}{c} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ \gamma_{xy}^T \end{array} \right\} = \left\{ \begin{array}{c} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{array} \right\} \Delta T \tag{5.58}$$

Using the plate strain displacement relations given by (5.6), the stress strain relations (5.55) in the $x-y$ coordinate system can be expressed as

ε° zero

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \left[\begin{array}{ccc} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{array} \right] \left(\left\{ \begin{array}{c} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{array} \right\} + z \left\{ \begin{array}{c} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right\} - \left\{ \begin{array}{c} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_{xy} \Delta T \end{array} \right\} \right) \tag{5.59}$$

Following the procedure in developing the plate theory, we obtain the plate constitutive equations as

$A^{-1} N^T = \varepsilon^T = \alpha \Delta T$

$$\left\{ \begin{array}{c} N \\ -- \\ M \end{array} \right\} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array} \right] \left\{ \begin{array}{c} \varepsilon^0 \\ -- \\ \kappa \end{array} \right\} - \left\{ \begin{array}{c} N^T \\ -- \\ M^T \end{array} \right\} \tag{5.60}$$

where

$$\left\{N^T\right\} = \left\{ \begin{array}{c} N_x^T \\ N_y^T \\ N_{xy}^T \end{array} \right\} = \int_{-h/2}^{h/2} \left[\overline{Q}\right] \left\{ \begin{array}{c} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{array} \right\} \Delta T dz \tag{5.61}$$

$N = A \varepsilon^{\circ}$

$\varepsilon^{\circ} = A^{-1} N$

$\{\sigma\} = [\overline{Q}](A^{-1}N - \{\alpha\}\Delta T\}$

$$\left\{M^T\right\} = \left\{ \begin{array}{c} M_x^T \\ \\ M_y^T \\ \\ M_{xy}^T \end{array} \right\} = \int_{-h/2}^{h/2} \left[\overline{Q}\right] \left\{ \begin{array}{c} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{array} \right\} \Delta T z dz \tag{5.62}$$

are thermal in plane and bending loads, respectively. After performing integration over the laminate thickness, the expressions in (5.61) and (5.62) can be written explicitly as

$$\left\{N^T\right\} = \Delta T \sum_{k=1}^{n} \left[\overline{Q}\right]_k \{\alpha\}_k t_k \tag{5.63a}$$

$$\left\{M^T\right\} = \frac{1}{2} \Delta T \sum_{k=1}^{n} \left[\overline{Q}\right]_k \{\alpha\}_k \left(z_k^2 - z_{k-1}^2\right) \tag{5.63b}$$

For free thermal expansions, no mechanical loads are applied, i.e., $\{N\} = \{M\} = 0$. We have

$$\left\{ \begin{array}{c} N^T \\ -- \\ M^T \end{array} \right\} = \left[\begin{array}{ccc} A & | & B \\ -- & -- & -- \\ B & | & D \end{array} \right] \left\{ \begin{array}{c} \varepsilon^{0T} \\ -- \\ \kappa^T \end{array} \right\} \tag{5.64}$$

This particular plate deformation is denoted by

$$\left\{ \begin{array}{c} \varepsilon^{0T} \\ -- \\ \kappa^T \end{array} \right\} = \left[\begin{array}{ccc} A & | & B \\ -- & -- & -- \\ B & | & D \end{array} \right]^{-1} \left\{ \begin{array}{c} N^T \\ -- \\ M^T \end{array} \right\} \tag{5.65}$$

For symmetric laminates, it is not difficult to see that $\{M^T\} = \{0\}$. Thus, with $[B] = [0]$, we obtain from (5.65)

$$\left\{ \begin{array}{c} N + N^T \\ \hline M + M^T \end{array} \right\} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array} \right] \left\{ \begin{array}{c} \varepsilon^{\circ} \\ \hline \kappa \end{array} \right\}$$

$$\{\varepsilon^{0T}\} = [A]^{-1}\{N^T\} \tag{5.66}$$

$$\{\kappa^T\} = \{0\}$$

This indicates that no out-of-plane deformation occurs in symmetric laminates when they experience a uniform temperature change.

Example 5.3 *Curing Stresses*

Epoxy based composite laminates are usually fabricated at elevated temperatures. The composite laminate experiences a drop in temperature during the cooling cycle. Due to the mismatch of thermal expansions, residual stresses (curing stresses) are often present.

Consider a $[\pm 45/0/90]_s$ laminate whose ply properties are given in Example 5.2. In addition, the thermal expansion coefficients are assumed to be $\alpha_1 = 2 \times 10^{-6}$ $\varepsilon/°\mathrm{F}$ and $\alpha_2 = 15 \times 10^{-6}$ $\varepsilon/°\mathrm{F}$. It is easy to see that

$$\begin{aligned}
\left\{\begin{array}{c}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{array}\right\}_{0°} &= \left\{\begin{array}{c}\alpha_1\\ \alpha_2\\ 0\end{array}\right\}\\
\left\{\begin{array}{c}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{array}\right\}_{90°} &= \left\{\begin{array}{c}\alpha_2\\ \alpha_1\\ 0\end{array}\right\}\\
\left\{\begin{array}{c}\alpha_x\\ \alpha_y\\ \alpha_{xy}\end{array}\right\}_{\pm 45°} &= \left\{\begin{array}{c}\frac{1}{2}(\alpha_1+\alpha_2)\\ \frac{1}{2}(\alpha_1+\alpha_2)\\ \pm(\alpha_1-\alpha_2)\end{array}\right\}
\end{aligned}$$

The matrices $[\overline{Q}]_{0°}$, $[\overline{Q}]_{90°}$, $[\overline{Q}]_{\pm 45°}$ and [A] are given in Example 5.2. Using (5.63) we obtain

$$\begin{aligned}
\left\{\begin{array}{c}N_x^T\\ \\ N_y^T\\ \\ N_{xy}^T\end{array}\right\} &= \left\{\begin{array}{c}1.37\\ 1.37\\ 0\end{array}\right\}\Delta T \text{ lb/in}\\
\{M^T\} &= \{0\}
\end{aligned}$$

In the absence of mechanical loads, {N} = {0} and {M} = {0} . It follows that

$$\begin{aligned}
\{\kappa\} &= \{0\}\\
\{\varepsilon\}_k &= \{\varepsilon^0\} = [A]^{-1}\{N^T\} \qquad \text{(a)}
\end{aligned}$$

The laminar stresses are obtained from (5.59), i.e.,

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\}_k = [\overline{Q}]_k \left\{\begin{array}{c}\varepsilon_x^0 - \Delta T\alpha_x\\ \varepsilon_y^0 - \Delta T\alpha_y\\ \gamma_{xy}^0 - \Delta T\alpha_{xy}\end{array}\right\} \qquad \text{(b)}$$

To estimate the stresses in the laminate after curing, the temperature change ΔT must first be determined. Since thermal expansion coefficients as well as elastic moduli are temperature-dependent, the evaluation of curing stresses using the present formulation is not precise. An approach to this problem is to use an effective temperature drop ΔT. For a 350°F cure epoxy system, the range of ΔT is usually taken between −250°F and −300°F.

We use $\Delta T = -250°F$ to estimate the curing stresses. The strains induced by this temperature drop are obtained from (a) as

$$\{\varepsilon^0\} = \begin{Bmatrix} -0.77 \\ -0.77 \\ 0 \end{Bmatrix} \text{x } 10^{-3}$$

The corresponding thermal residual stresses are obtained from (b). We have

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_{0°} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{0°} = \begin{Bmatrix} -4092 \\ 4092 \\ 0 \end{Bmatrix} \text{psi}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_{90°} = \begin{Bmatrix} \sigma_{22} \\ \sigma_{11} \\ \sigma_{12} \end{Bmatrix}_{90°} = \begin{Bmatrix} 4092 \\ -4092 \\ 0 \end{Bmatrix} \text{psi}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_{\pm 45°} = \begin{Bmatrix} 0 \\ 0 \\ \mp 4092 \end{Bmatrix} \text{psi}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_{\pm 45°} = \begin{Bmatrix} -4092 \\ 4092 \\ 0 \end{Bmatrix} \text{psi}$$

The curing stresses can be quite significant as compared with the transverse strength of the unidirectional composite.

ΔT

5.9 TAILORING OF LAMINATE THERMAL EXPANSION $\{\bar\alpha\} \neq \{\alpha\}$

$\varepsilon^T = \Delta T \cdot \alpha$

One of the unique characteristics provided by composite laminates is the tailorability of their thermal dimensions. Since many fibers such as carbon and Kevlar fibers possess negative thermal expansion coefficients, it is conceivable that composite structures with near zero thermal expansions can be realized. ($\bar\alpha$ depends on thermal expansion & material elastic property of each lamina)

α: is a property of the material

The *effective* laminate linear thermal expansion of a symmetric laminate can be represented by the average thermal expansion coefficients defined as

$\{N^T\} = [A]\{\varepsilon^{0T}\}$; $N = 0$

$\to \{\alpha\} = \{\varepsilon^{0T}\}/\Delta T = [A]^{-1}\{N^T\}/\Delta T$

in laminates, α can be tailored

$$\boxed{\{\bar\alpha\} = \frac{1}{\Delta T}\{\varepsilon^{0T}\} = [A]^{-1}(N^T/\Delta T)}$$

$$= [A]^{-1}\int_{-h/2}^{h/2}[\overline{Q}]\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} dz = [A]^{-1}\sum_{k=1}^{n}[\overline{Q}]_k \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_k t_k \quad (5.67)$$

integrated layer by layer

In deriving (5.67), (5.61) and (5.66) have been used. From (5.67), it is evident that the effective laminate thermal expansion coefficients $\{\bar\alpha\}$ depend on the lamina thermal expansion coefficients as well as laminate stiffnesses.

Experimentally, Implant T, measure ε^T then $\alpha = \dfrac{\varepsilon^T}{\Delta T}$

for an non homogeneous solid. $\varepsilon^T \neq \Delta T \cdot \alpha$,

↳ then, need to find $\bar\alpha_{average}$. $\varepsilon^T = \Delta T\,\bar\alpha$. $\therefore\ \alpha = \dfrac{\varepsilon^T}{\Delta T}$

by changing orientation, $\bar{Q}$ changes, which changes $\bar{\alpha}$

Consider a $[0/90]_s$ laminate. After some straightforward manipulations, we obtain

$$\left\{ \begin{array}{c} \overline{\alpha}_x \\ \overline{\alpha}_y \\ \overline{\alpha}_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \overline{\alpha}_0 \\ \overline{\alpha}_0 \\ 0 \end{array} \right\} \tag{5.68}$$

where

$$\overline{\alpha}_0 = \frac{\alpha_1(Q_{11}+Q_{12}) + \alpha_2(Q_{12}+Q_{22})}{Q_{11}+Q_{22}+2Q_{12}} \tag{5.69}$$

From the coordinate transformation equations for thermal expansion coefficients, (5.56) or (5.57), it is easy to verify that a material having thermal expansion coefficients of the form given by (5.69) is isotropic in thermal expansion. In fact, this is true for all cross-plied laminates with the same number of 0° and 90° plies.

To achieve an in-plane zero thermal expansion in the $[0/90]_S$ laminate, it is necessary that $\overline{\alpha}_0 = 0$, which leads to the condition

$$\alpha_1 = -\frac{Q_{12}+Q_{22}}{Q_{12}+Q_{11}}\alpha_2 = -\frac{1+\nu_{12}}{\dfrac{E_1}{E_2}+\nu_{12}}\alpha_2 \tag{5.70}$$

For laminates with more than two fiber orientations, it can be shown that quasi-isotropic laminates also possess isotropic in-plane thermal expansion, and the coefficient of the isotropic thermal expansion has the same expression as given in (5.69). The proof can be carried out by noting that

$$\begin{aligned} \left\{N^T/\Delta T\right\} &= \sum_{k=1}^{N} \left[\overline{Q}\right]_k \{\alpha\}_k t_k \\ &= \sum_{k=1}^{N} [T_\epsilon]_k^T [Q] [T_\epsilon]_k \left\{ \begin{array}{c} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{array} \right\}_k t_k \\ &= \left(\sum_{k=1}^{N} [T_\epsilon]_k^T\right) [Q] \left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ 0 \end{array} \right\} t \end{aligned} \tag{5.71}$$

where N is the number of fiber orientations in the laminate, and t = h/N is the total thickness of all the laminae with the same fiber orientation. In the last step of (5.71), the relation (5.56) has been used.

Using the results of (5.35) with N $\geq$ 3 and $\theta_k = k\pi/N$, we have

$$\begin{aligned} \sum_{k=1}^{N} \cos^2\theta_k &= \frac{N}{2} + \sum_{k=1}^{n} \cos 2\theta_k = \frac{N}{2} \\ \sum_{k=1}^{N} \sin^2\theta_k &= \frac{N}{2} - \sum_{k=1}^{n} \cos 2\theta_k = \frac{N}{2} \end{aligned} \tag{5.72}$$

$$\sum_{k=1}^{N} \cos\theta_k \sin\theta_k = \frac{1}{2}\sum_{k=1}^{n} \sin 2\theta_k = 0$$

With the above relations, one can easily verify the following

$$\sum_{k=1}^{N} [T_\varepsilon]_k^T = \frac{N}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.73}$$

For quasi-isotropic laminates, A_{ij} are given by (5.40), i.e.,

$$[A] = h\begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & U_5 \end{bmatrix} \tag{5.74}$$

Thus, the average coefficients of thermal expansion can be expressed as

$$\left\{ \begin{array}{c} \overline{\alpha}_x \\ \overline{\alpha}_y \\ \overline{\alpha}_{xy} \end{array} \right\} = [A]^{-1}\left\{N^T/\Delta T\right\} = [A]^{-1}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}[Q]\left\{ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ 0 \end{array} \right\}\frac{h}{2} \tag{5.75}$$

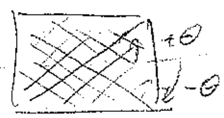
which can be shown to be identical to (5.68) after performing some multiplication.

Example 5.4 *Negative Thermal Expansion Coefficient*

As indicated by (5.67), the effective (average) coefficients of thermal expansion of a laminate also depend on the stiffnesses of the laminae. Consider the $[\pm\theta]_S$ laminates of a graphite/epoxy composite whose material constants are

$$\begin{aligned} E_1 &= 20\times 10^6 psi \quad , \quad E_2 = 1.45\times 10^6 psi \\ G_{12} &= 1.0\times 10^6 psi \quad , \quad \nu_{12} = 0.3 \\ \alpha_1 &= 1\times 10^{-6}/^\circ F \quad , \quad \alpha_2 = 20\times 10^6/^\circ F \end{aligned}$$

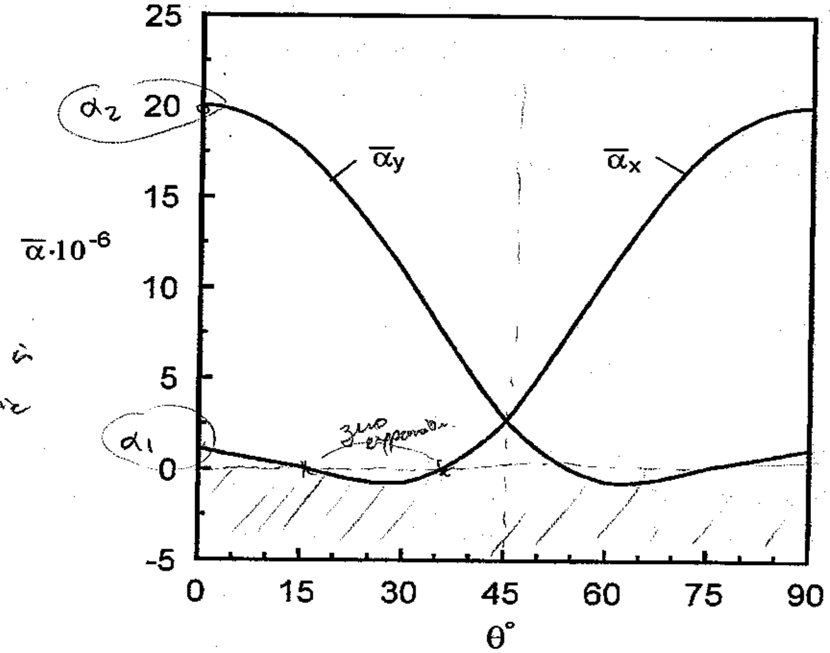
The effective coefficients of thermal expansion $\overline{\alpha}_x$ and $\overline{\alpha}_y$ are computed using (5.67). Figure 5.8 shows the variation of $\overline{\alpha}_x$ and $\overline{\alpha}_y$ as a function of θ. It is interesting to note that negative thermal expansion can be produced by selecting a θ that falls within 16° and 37°.



$\alpha_1 = 1 \times 10^{-6}/°F$

$\alpha_2 = 20 \times 10^{-6}/°F$

Want
Isotropic laminate in terms of elastic (in-plane) properties.
$N \geq 3$.



or negative thermal expansion coefficient.

Figure 5.8 Effective thermal expansion coefficients of $[\pm\theta]$ graphite/epoxy laminates

5.10 PLATE EQUATIONS OF MOTION (Equilibrium equations for laminates)

The 3-D equations of motion at any point in the laminate can be obtained from the equations of equilibrium, (2.50), by setting the body force equal to the inertia force. Using the approximate plate displacements (5.5), the equations of motion are

$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^\circ \\ \kappa \end{Bmatrix}$

need governing equations to tie N, M & $\varepsilon^\circ, \kappa$

In 3D solids.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} = \rho\left(\ddot{u}_o - z\frac{\partial \ddot{w}_o}{\partial x}\right) \quad \Sigma F_x = 0 \quad (5.76)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2} = \rho\left(\ddot{v}_o - z\frac{\partial \ddot{w}_o}{\partial y}\right) \quad \Sigma F_y = 0 \quad (5.77)$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} = \rho \ddot{w}_o \quad \Sigma F_z = 0 \quad (5.78)$$

on a cubic element

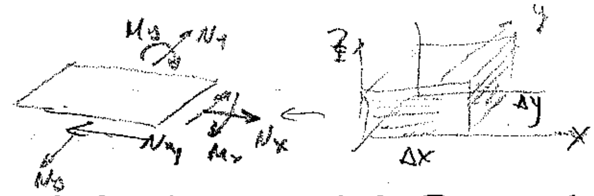
must be satisfied in the laminate

where a dot indicates time derivative.

From the theory of elasticity, we recall that an arbitrarily assumed stress field may not satisfy the compatibility equations, while an arbitrarily assumed displacement field may not satisfy the equilibrium equations. Therefore, the stresses computed from the approximate plate displacements must be handled with care. In view of the foregoing, the full 3-D stress components are included in deriving the plate equations of motion.

One of the methods used for deriving the plate equations of motion is the integration of the 3-D equations of motion over the plate thickness. Specifically, equations (5.76-5.78) are first integrated over the laminate thickness to ensure the

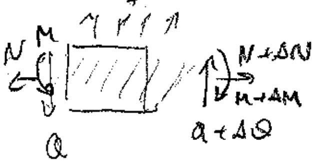
3D Equilibrium on laminate:



global balance of forces in the x, y, and $z$directions, respectively. To ensure balance of moments about x and y axes, equations (5.76) and (5.77) are multiplied by z and then integrated over the plate thicknesses, respectively. Thus, five plate equations of motion are obtained.

Beam eqn.:

To illustrate this procedure, we now proceed to integrate (5.76) with respect to z:



$$\int_{-h/2}^{h/2} \frac{\partial \sigma_{xx}}{\partial x} dz + \int_{-h/2}^{h/2} \frac{\partial \sigma_{xy}}{\partial y} dz + \int_{-h/2}^{h/2} \frac{\partial \sigma_{xz}}{\partial z} dz = \int_{-h/2}^{h/2} \rho \left(\ddot{u}_0 - z \frac{\partial \ddot{w}_0}{\partial x} \right) dz \tag{5.79}$$

$\Sigma F_x = 0$

$\Delta N = 0$

$\boxed{N = \text{constant}}$

Assume that the laminated plate is subjected to only transverse tractions on the top and bottom surfaces; $z = \pm\, h/2$ and the third integral on the left hand side of (5.79) vanishes. In terms of the plate resultant forces, (5.79) can be written as

$\Sigma F_y = 0$

$(Q + dQ) - Q + q\Delta x = 0$

$\frac{\Delta Q}{\Delta x} + q = 0$

$\boxed{\frac{dQ}{dx} + q = 0}$

$$\boxed{\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = P\ddot{u}_o - R\frac{\partial \ddot{w}_0}{\partial x} = 0} \quad (1') \tag{5.80}$$

$\Sigma M_x = 0$

$\int_{-h/2}^{h/2} (1)\, z \cdot dz$

in which

$$P = \int_{-h/2}^{h/2} \rho dz$$

$$R = \int_{-h/2}^{h/2} \rho z dz$$

$\Sigma M_y = 0$

$\int_{-h/2}^{h/2} (2)\, z\, dz$

$M = \int_{-h/2}^{h/2} \sigma_{xx} \cdot y \, dA$

$\Sigma M_x = 0$

$(M + \Delta M) - M + (Q + \Delta Q)\Delta x + q \cdot \Delta x \cdot \frac{1}{2}\Delta x = 0$

$\Delta M + Q\Delta x + \cancel{\Delta Q \Delta x} + \frac{1}{2} \cancel{q \Delta x^2} = 0$

$\frac{\Delta M}{\Delta x} = -Q$

$\frac{dM}{dx} = -Q$

$\Rightarrow \boxed{\frac{d^2M}{dx^2} = q}$

It is obvious that when the laminae have the same mass density (e.g., all laminae are of the same composite), R vanishes automatically.

In a similar manner, (5.77) can be integrated with the result

$$\boxed{\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = Pv_o - R\frac{\partial \ddot{w}_0}{\partial y} = 0} \quad (2') \tag{5.81}$$

Integration of (5.78) yields

$$\boxed{\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = P\ddot{w}_0 = 0} \quad (3') \tag{5.82}$$

where

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (\sigma_{xz}, \sigma_{yz}) dz$$

and

$$q = \sigma_{zz}\left(\frac{h}{2}\right) - \sigma_{zz}\left(\frac{-h}{2}\right) = q^+ - q^-$$

The quantities Q_x and Q_y are the shear forces acting on the x face and y face, respectively; q is the net applied transverse load (N/m^2), and q^+ and q^- denote the transverse tractions on the top and bottom surfaces, respectively.

Multiplying (5.76) by z and integrating with respect to z over the plate thickness yield

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \int_{-h/2}^{h/2} \frac{\partial \sigma_{xz}}{\partial z} z dz = \int_{-h/2}^{h/2} \left(\rho \ddot{u}_o z - \rho \frac{\partial \ddot{w}_o}{\partial x} z^2\right) dz \tag{5.83}$$

Noting that

$$\frac{\partial \sigma_{xz}}{\partial z} z = \frac{\partial (z\sigma_{xz})}{\partial z} - \sigma_{xz}$$

we have

$$\int_{-h/2}^{h/2} \frac{\partial \sigma_{xz}}{\partial z} z dz = z\sigma_{xz}\,\Big|_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \sigma_{xz} dz$$

Since there is no in-plane shear loading on the top and bottom surfaces of the plate, the first term on the right hand side of the above equation must vanish. Thus,

$$\int_{-h/2}^{h/2} \frac{\partial \sigma_{xz}}{\partial z} z dz = -Q_x$$

and (5.83) becomes

$$\boxed{\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = R\ddot{u}_o - I\frac{\partial \ddot{w}_o}{\partial x} = 0} \quad (4') \tag{5.84}$$

where

$$I = \int_{-h/2}^{h/2} \rho z^2 dz$$

is the mass moment of inertia.

A similar procedure on (5.77) leads to

$$\boxed{\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = R\ddot{v}_0 - I\frac{\partial \ddot{w}_0}{\partial y} = 0} \quad (5') \tag{5.85}$$

Eliminating Q_x and Q_y from (5.82) using (5.84) and (5.85) yields

$$\boxed{\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = P\ddot{w}_0 + R\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) - I\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}\right)} \tag{5.86}$$

Equations (5.80), (5.81), and (5.86) are the plate equations of motion in terms of the resultant forces and moments. These equations can also be expressed in plate displacement components u_0, v_0, and w_0. From the plate constitutive relations, the resultant forces and moments can be written in terms of the plate displacements u_0, v_0, and w_0 as

It is desirable to express this equation in terms of U_o, V_o, W_o

$$\begin{aligned}
N_x &= A_{11}\frac{\partial u_0}{\partial x} + A_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + A_{12}\frac{\partial v_0}{\partial y} - B_{11}\frac{\partial^2 w_0}{\partial x^2} \\
&\quad -2B_{16}\frac{\partial^2 w_0}{\partial x \partial y} - B_{12}\frac{\partial^2 w_0}{\partial y^2} \\
N_y &= A_{12}\frac{\partial u_0}{\partial x} + A_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + A_{22}\frac{\partial v_0}{\partial y} - B_{12}\frac{\partial^2 w_0}{\partial x^2} \\
&\quad -2B_{26}\frac{\partial^2 w_0}{\partial x \partial y} - B_{22}\frac{\partial^2 w_0}{\partial y^2} \\
N_{xy} &= A_{16}\frac{\partial u_0}{\partial x} + A_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + A_{26}\frac{\partial v_0}{\partial y} - B_{16}\frac{\partial^2 w_0}{\partial x^2} \\
&\quad -2B_{66}\frac{\partial w_0}{\partial x \partial y} - B_{26}\frac{\partial^2 w_0}{\partial y^2} \\
M_x &= B_{11}\frac{\partial u_0}{\partial x} + B_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + B_{12}\frac{\partial v_0}{\partial y} - D_{11}\frac{\partial^2 w_0}{\partial x^2} \\
&\quad -2D_{16}\frac{\partial^2 w_0}{\partial x \partial y} - D_{12}\frac{\partial^2 w_0}{\partial y^2} \\
M_y &= B_{12}\frac{\partial u_0}{\partial x} + B_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + B_{22}\frac{\partial v_0}{\partial y} - D_{12}\frac{\partial^2 w_0}{\partial x^2} \\
&\quad -2D_{26}\frac{\partial^2 w_0}{\partial x \partial y} - D_{22}\frac{\partial^2 w_0}{\partial y^2} \\
M_{xy} &= B_{16}\frac{\partial u_0}{\partial x} + B_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + B_{26}\frac{\partial v_0}{\partial y} - D_{16}\frac{\partial^2 w_0}{\partial x^2} \\
&\quad -2D_{66}\frac{\partial^2 w_0}{\partial x \partial y} - D_{26}\frac{\partial^2 w_0}{\partial y^2}
\end{aligned} \tag{5.87}$$

$$\begin{Bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\varepsilon^0} \\ \kappa \end{Bmatrix}$$

Substituting the above expressions into the plate equations of motion, (5.80), (5.81) and (5.86), we obtain the plate equations of motion in displacements:

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + 2A_{16}\frac{\partial^2 u_0}{\partial x \partial y} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 v_0}{\partial x \partial y} + A_{26}\frac{\partial^2 v_0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w_0}{\partial x^3} - 3B_{16}\frac{\partial^3 w_0}{\partial x^2 \partial y} - (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x \partial y^2}$$

$$-B_{26}\frac{\partial^3 w_o}{\partial y^3} = P\ddot{u}_o - R\frac{\partial \ddot{w}_o}{\partial x} \tag{5.88a}$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2} + (A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x \partial y} + A_{26}\frac{\partial^2 u_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + 2A_{26}\frac{\partial^2 v_0}{\partial x \partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2}$$

$$-B_{16}\frac{\partial^3 w_o}{\partial x^3} - (B_{12}+2B_{66})\frac{\partial^3 w_o}{\partial x^2 \partial y} - 3B_{26}\frac{\partial^3 w_0}{\partial x \partial y^2}$$

$$-B_{22}\frac{\partial^3 w_o}{\partial y^3} = P\ddot{v}_o - R\frac{\partial \ddot{w}_o}{\partial y} \tag{5.88b}$$

$$D_{11}\frac{\partial^4 w_o}{\partial x^4} + 4D_{16}\frac{\partial^4 w_o}{\partial x^3 \partial y} + 2(D_{12}+2D_{66})\frac{\partial^4 w_o}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w_o}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w_o}{\partial y^4}$$

$$-B_{11}\frac{\partial^3 u_0}{\partial x^3} - 3B_{16}\frac{\partial^3 u_0}{\partial x^2 \partial y} - (B_{12}+2B_{66})\frac{\partial^3 u_0}{\partial x \partial y^2} - B_{26}\frac{\partial^3 u_0}{\partial y^3} - B_{16}\frac{\partial^3 v_0}{\partial x^3}$$

$$-(B_{12}+2B_{66})\frac{\partial^3 v_0}{\partial x^2 \partial y} - 3B_{26}\frac{\partial^3 v_0}{\partial x \partial y^2} - B_{22}\frac{\partial^3 v_0}{\partial y^3}$$

$$+P\ddot{w}_o + R\left(\frac{\partial \ddot{u}_o}{\partial x} + \frac{\partial \ddot{v}_o}{\partial y}\right) - I\left(\frac{\partial^2 \ddot{w}_o}{\partial x^2} + \frac{\partial^2 \ddot{w}_o}{\partial y^2}\right) = q \tag{5.88c}$$

Symmetric Laminates

For thin laminates undergoing transverse deflections, the mass coupling inertia coefficient R and the rotatory inertia I are usually negligible and can be set equal to zero. In addition, if the laminate possesses a symmetric stacking sequence, then $B_{ij} = 0$ and the equations of motion are simplified to

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + 2A_{16}\frac{\partial^2 u_0}{\partial x \partial y} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + A_{16}\frac{\partial^2 u_0}{\partial x^2}$$

$$+(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x \partial y} + A_{26}\frac{\partial^2 v_0}{\partial y^2} = P\ddot{u}_0 \tag{5.89a}$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2} + (A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x \partial y} + A_{26}\frac{\partial^2 u_0}{\partial y^2}$$

$$+A_{66}\frac{\partial^2 v_0}{\partial x^2} + 2A_{26}\frac{\partial^2 v_0}{\partial x \partial y} + A_{22}\frac{\partial^2 v_0}{\partial y^2} = P\ddot{v}_0 \tag{5.89b}$$

$$D_{11}\frac{\partial^4 w_0}{\partial x^4}+4D_{16}\frac{\partial^4 w_0}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w_0}{\partial x^2\partial y^2}$$
$$+4D_{26}\frac{\partial^4 w_0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w_0}{\partial y^4}+P\ddot{w}_0=q \tag{5.89c}$$

It is noted that, for symmetric laminates, the in-plane motion and the flexural motion are uncoupled.

For an isotropic plate, $D_{16}=D_{26}=0$, and

$$\begin{aligned} D_{11} &= D_{22}=D=\frac{Eh^3}{12(1-\nu^2)} \\ D_{12} &= \nu D \\ D_{66} &= \frac{(1-\nu)}{2}D \end{aligned}$$

and the transverse plate equation of motion reduces to the well known classical plate equation

$$D\,\nabla^4\,w_o+P\ddot{w}_0=q \tag{5.90}$$

where

$$\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$$

5.11 BOUNDARY CONDITIONS

Solutions to the plate equations of equilibrium are completed by imposing proper boundary conditions. Depending on the constraint conditions along the plate boundary, either displacements and the slope or resultant forces and moments are prescribed. At the plate boundary, the in plane plate displacements can be given by u_o and v_o or by the normal (to the edge) component u_n and the tangential component u_s, see Fig. 5.9. Similarly, the resultant forces and moments can also be decomposed with respect to the normal (n) and tangential (s) directions. The following boundary conditions are written in terms of these quantities.

- Simply-supported-movable in the plane of the plate

$$N_n=N_{ns}=w_o=M_n=0 \tag{5.91}$$

- Hinged-immovable in the plane of the plate

$$u_n=u_s=w_o=M_n=0 \tag{5.92}$$

- Hinged-free in the normal direction

$$N_n=u_s=w_o=M_n=0 \tag{5.93}$$

- Hinged-free in the tangential direction

$$u_n = N_{ns} = w_o = M_n = 0 \tag{5.94}$$

- Clamped

$$u_n = u_s = w_o = \frac{\partial w_o}{\partial n} = 0 \tag{5.95}$$

- Free

$$N_n = N_{ns} = M_n = \frac{M_{ns}}{\partial s} + Q_n = 0 \tag{5.96}$$

The displacement components u_n and u_s follow the coordinate transformation of the components of a vector, i.e.,

$$\begin{aligned} u_n &= u_0 \text{cos}\alpha + v_0 \text{sin}\alpha \\ u_s &= u_0 \text{sin}\alpha - v_0 \text{cos}\alpha \end{aligned} \tag{5.97}$$

where α is the angle between the x axis and the normal direction to the boundary contour (see Fig. 5.9).

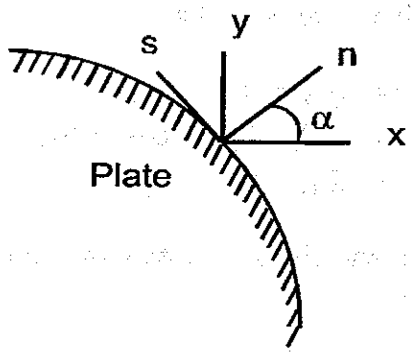


Figure 5.9

The resultant forces N_n and N_{ns} and the resultant moments M_n and M_{ns} follow the coordinate transformation of the components of the stress tensor. We have

$$\begin{aligned} N_n &= N_x \cos^2\alpha + N_y \text{sin}^2\alpha + 2N_{xy}\text{sin}\alpha\ \text{cos}\alpha \\ N_{ns} &= N_{xy}(\cos^2\alpha - \text{sin}^2\alpha) + (N_y - N_x)\text{sin}\alpha\ \text{cos}\alpha \\ M_n &= M_x \cos^2\alpha + M_y \text{sin}^2\alpha + 2M_{xy}\text{sin}\alpha\ \text{cos}\alpha \\ M_{ns} &= M_{xy}(\cos^2\alpha - \text{sin}^2\alpha) + (M_y - M_x)\text{sin}\alpha\ \text{cos}\alpha \end{aligned} \tag{5.98}$$

Note that in the free edge boundary conditions (5.96) the last condition

$$\frac{\partial M_{ns}}{\partial s} + Q_n = 0$$

is used instead of $M_{ns} = Q_n = 0$. The explanation can be found in Timoshenko and Woinowsky-Krieger [5.3].

Example 5.5 *Bending of Unsymmetrical Laminates*

For unsymmetric laminates the B matrix is not null, and bending-extension coupling exists. This example serves to illustrate such a coupling effect.

Consider a [0/90] laminate hinged at both edges as shown in Fig. 5.10. The laminate is subjected to a uniform transverse load q. Assume that the laminate is very long in the y-direction such that the deformation can be approximated by cylindrical bending, i.e., u_0, v_0 and w_0 are functions of x only. For this particular case, v_0 is not coupled with u_0 and w_0 and is set equal to zero.

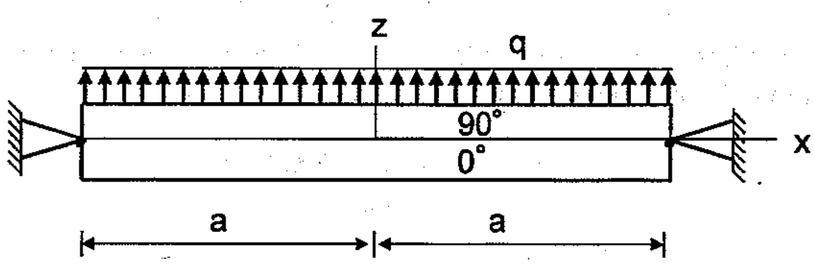


Figure 5.10 Hinged-hinged laminate under uniform transvese load

For the [0/90] laminate we note that

$$\begin{aligned} A_{16} &= A_{26} = 0 \\ B_{22} &= -B_{11}, B_{12} = B_{66} = B_{16} = B_{26} = 0 \\ D_{16} &= D_{26} = 0 \end{aligned}$$

Using the above properties and the assumption of cylindrical bending, the governing equations (5.87) reduce to

5.88

$$A_{11}\frac{d^2u_0}{dx^2} - B_{11}\frac{d^3w_o}{dx^3} = 0 \qquad \text{(a)}$$

$$D_{11}\frac{d^4w_o}{dx^4} - B_{11}\frac{d^3u_0}{dx^3} = q \qquad \text{(b)}$$

Using (a) to eliminate u_0 from (b), we obtain

$$\frac{d^4w_o}{dx^4} = \frac{q}{D} \qquad \text{(c)}$$

where

$$D = D_{11} - \frac{B_{11}^2}{A_{11}}$$

The general solution for the ordinary differential equation given by (c) is easily obtained as

$$w_o(x) = \frac{q}{24D}x^4 + C_1x^3 + C_2x^2 + C_3x + C_4 \qquad \text{(d)}$$

Since the deflection should be symmetric with respect to the center, we readily conclude that

$$C_1 = C_3 = 0$$

From (5.87) we note that, for cylindrical bending,

$$N_x = A_{11}\frac{du_o}{dx} - B_{11}\frac{d^2w_o}{dx^2} \tag{e}$$

Comparing (e) with (a), we conclude that $\frac{dN_x}{dx} = 0$ and thus

$$N_x = \text{constant} = N_x^o$$

This implies that the in-plane force in the laminate is uniform. From (e) we have

$$\frac{du_0}{dx} = \frac{1}{A_{11}}N_x^0 + \frac{B_{11}}{A_{11}}\frac{d^2w_0}{dx^2}$$

Integrating the above equation and recognizing $u_0\ (0) = 0$ and $\frac{dw_0}{dx} = 0$ at $x = 0$ because of symmetry, we obtain

$$u_0(x) = \frac{1}{A_{11}}N_x^0 x + \frac{B_{11}}{A_{11}}\frac{dw_0}{dx} \tag{f}$$

The boundary conditions are

$$u_0(a) = 0, w_0(a) = 0, M_x(a) = 0 \tag{g}$$

The moment free boundary condition can be expressed using the following relation

$$\begin{aligned} M_x &= B_{11}\frac{du_o}{dx} - D_{11}\frac{d^2w_o}{dx^2} \\ &= \frac{B_{11}}{A_{11}}N_x^0 - \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\frac{d^2w_o}{dx^2} \end{aligned}$$

Using the general solutions given by (d) and (f), the boundary conditions become

$$\begin{aligned} N_x^0 + \frac{B_{11}q}{6D}a^2 + 2B_{11}C_2 &= 0 \\ \frac{q}{24D}a^4 + a^2C_2 + C_4 &= 0 \\ \frac{B_{11}}{A_{11}}N_x^0 - \frac{1}{2}qa^2 + 2DC_2 &= 0 \end{aligned}$$

Solving the above equations, we obtain

$$\begin{aligned} C_2 &= -\frac{3A_{11}D + B_{11}^2}{12A_{11}D_{11}D}qa^2 \\ C_4 &= -a^2C_2 - \frac{q}{24D}a^4 \\ N_x^0 &= 2B_{11}C_2 - \frac{B_{11}}{6D}qa^2 \end{aligned}$$

With these determined coefficients, the deflection is obtained from (d) as

$$w_0(x) = \frac{q}{24D}(x^4 - a^4) - \frac{3A_{11}D + B_{11}^2}{12A_{11}D_{11}D}qa^2(x^2 - a^2)$$

The maximum deflection occurs at $x = 0$.

$$\begin{aligned} (w_0)_{max} &= -\frac{qa^4}{24D} + \frac{3A_{11}D + B_{11}^2}{12A_{11}D_{11}D}qa^4 \\ &= \left[\frac{5}{24D_{11}} + \frac{B_{11}^2}{24D_{11}(D_{11}A_{11} - B_{11}^2)}\right]qa^4 \end{aligned} \tag{h}$$

The constant in plane force can be simplified and expressed as

$$N_x^0 = \frac{B_{11}}{3D_{11}}qa^2 \tag{i}$$

It is obvious from (h) that extension-bending coupling increases the deflection and thus reduces the effective bending stiffness of the laminate Also note that increasing bending stiffness D_{11} has the same effect as reducing extension-bending coupling. It is interesting to note from (i) that the in-plane force N_x is proportional to B_{11}. For [0/90], B_{11} is negative, and for [90/0], B_{11} is positive . Thus, for a positive ($q > 0$) load, the [0/90] laminate would be under in-plane compression, while the [90/0] laminate would be under in-plane tension. Further discussion on bending of unsymmetric laminates is given in Chapter 7.

5.12 MINDLIN PLATE THEORY

In the previous sections, classical plate theory (CPT) was developed based on two term displacement expansions given by (5.2). In addition, the transverse shear deformations γ_{xz} and γ_{yz} were suppressed. As a consequence, rotations ψ_x and ψ_y are equal to the slopes $-\partial w_0/\partial x$ and $-\partial w_0/\partial y$, respectively, and only three independent kinematic variables u_0, v_0 and w_0 remain.

For laminates composed of fiber composites, the in-plane rigidity is usually much greater than the transverse shear rigidity, and neglecting the transverse shear deformation may result in appreciable errors, especially when the plate thickness is not small (to be discussed later) as compared to the characteristic length (wave length) of deformation. A more accurate plate theory can be developed by allowing transverse shear strains γ_{xz} and γ_{yz} and considering u_0, v_0, w_0, ψ_x and ψ_y as independent kinematic variables. Such an approach was used by Mindlin [5.4] to develop a shear-deformable theory for plates of isotropic materials. A similar theory was developed by Reissner [5.5]. Later, several authors [5.6-5.7] extended Mindlin's theory to composite laminates. The governing equations of the Mindlin plate theory for composite laminates can be derived in exactly the same manner as discussed in the previous sections. Results for the isothermal case are summarized as follows.

Plate Constitutive Equations

In-plane forces and bending moments:

$$\left\{\begin{array}{c} N \\ -- \\ M \end{array}\right\} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]\left\{\begin{array}{c} \varepsilon^0 \\ -- \\ \kappa \end{array}\right\}^{-1} \tag{5.99}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \qquad \varepsilon_y^0 = \frac{\partial v_0}{\partial y}, \qquad \gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$

$$\kappa_x = \frac{\partial \psi_x}{\partial x}, \qquad \kappa_y = \frac{\partial \psi_y}{\partial y}, \qquad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \tag{5.100}$$

The resultant forces N_x, N_y, N_{xy}, the bending moments M_x, M_y, M_{xy}, and A, B, D matrices are the same as in classical plate theory.

Transverse shear forces:

$$\left\{\begin{array}{c} Q_y \\ \\ Q_x \end{array}\right\} = k\left[\begin{array}{cc} A_{44} & A_{45} \\ \\ A_{45} & A_{55} \end{array}\right]\left\{\begin{array}{c} \gamma_{yz}^0 \\ \\ \gamma_{xz}^0 \end{array}\right\} \tag{5.101}$$

where

$$\gamma_{yz}^0 = \frac{\partial w_0}{\partial y} + \psi_y \qquad , \qquad \gamma_{xz}^0 = \frac{\partial w_0}{\partial x} + \psi_x \tag{5.102}$$

are the transverse shear strains (assumed constant over the plate thickness) in the $y-z$ and $x-z$ planes, respectively; k is a shear correction coefficient introduced to compensate for the error resulting from the constant shear strain assumption; and A_{ij} $(i,j= 4{,}5)$ are the plate transverse shear stiffnesses defined as

$$A_{ij} = \int_{-h/2}^{h/2} C_{ij}dz \qquad i,j = 4,5 \tag{5.103}$$

In (5.104), C_{ij} are the transformed elastic constants in reference to the x-y-z coordinate system.

For plates of isotropic materials, various ways were suggested to determine the value of the shear correction coefficient k. Basically, k was chosen to match certain exact elasticity solutions. For example, Mindlin [5.4] matched the plate solution with the exact elasticity solution for the natural frequency of the first antisymmetric mode of thickness shear vibration (such as $\psi_x = e^{i\omega t}, \psi_y = w_0 = 0$) and obtained $k = \pi^2/12$. On the other hand, Reissner [5.5] obtained k = 5/6 based on consideration of the exact transverse shear stress distribution. For general composite laminates with many possibilities of different stacking sequences, the above methods cannot yield a unique value of k. Based on a numerical study of some laminates, Whitney and Pagano [5.7] found that k = 5/6 was suitable for composite plates.

Equations of Motion

In terms of resultant forces and bending moments, the plate equations of motion are given by

$$\begin{aligned}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= P\ddot{u}_0 + R\ddot{\psi}_x \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= P\ddot{v}_0 + R\ddot{\psi}_y \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q &= P\ddot{w}_0 \\
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= R\ddot{u}_0 + I\ddot{\psi}_x \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= R\ddot{v}_0 + I\ddot{\psi}_y
\end{aligned} \tag{5.104}$$

Substitution of the plate constitutive equations (5.100-5.102) in (5.104) yields the displacement equations of motion.

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + 2A_{16}\frac{\partial^2 u_0}{\partial x \partial y} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + A_{16}\frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 v_0}{\partial x \partial y}$$

$$+A_{26}\frac{\partial^2 v_0}{\partial y^2} + B_{11}\frac{\partial^2 \psi_x}{\partial x^2} + 2B_{16}\frac{\partial^2 \psi_x}{\partial x \partial y} + B_{66}\frac{\partial^2 \psi_x}{\partial y^2} + B_{16}\frac{\partial^2 \psi_y}{\partial x^2}$$

$$+(B_{12} + B_{66})\frac{\partial^2 \psi_y}{\partial x \partial y} + B_{26}\frac{\partial^2 \psi_y}{\partial y^2} = P\ddot{u}_0 - R\ddot{\psi}_x$$

$$A_{16}\frac{\partial^2 u_0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 u_0}{\partial x \partial y} + A_{26}\frac{\partial^2 u_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + 2A_{26}\frac{\partial^2 v_0}{\partial x \partial y}$$

$$+A_{22}\frac{\partial^2 v_0}{\partial y^2} + B_{16}\frac{\partial^2 \psi_x}{\partial x^2} + (B_{12}+B_{66})\frac{\partial_x^2 \psi_x}{\partial x \partial y} + B_{26}\frac{\partial^2 \psi_x}{\partial x \partial y}$$

$$B_{66}\frac{\partial^2 \psi_y}{\partial x^2} + B_{26}\frac{\partial^2 \psi_y}{\partial x \partial y} + B_{22}\frac{\partial^2 \psi_y}{\partial y^2} = P\ddot{v}_0 + R\ddot{\psi}_y$$

$$kA_{55}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) + kA_{45}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x \partial y}\right)$$

$$+kA_{44}\left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) + q = P\ddot{w}_0 \tag{5.105}$$

$$B_{11}\frac{\partial^2 u_0}{\partial x^2} + 2B_{16}\frac{\partial^2 u_0}{\partial x \partial y} + B_{66}\frac{\partial^2 u_0}{\partial y^2} + B_{16}\frac{\partial^2 v_0}{\partial x^2} + (B_{12}+B_{66})\frac{\partial^2 v_0}{\partial x \partial y} + B_{26}\frac{\partial^2 v_0}{\partial y^2}$$

$$+D_{11}\frac{\partial^2 \psi_x}{\partial x^2} + 2D_{16}\frac{\partial^2 \psi_x}{\partial x \partial y} + D_{66}\frac{\partial^2 \psi_x}{\partial y^2} + D_{16}\frac{\partial^2 \psi_y}{\partial x^2} + (D_{12}+D_{66})\frac{\partial^2 \psi_y}{\partial x \partial y}$$

$$+D_{26}\frac{\partial^2 \psi_y}{\partial y^2} - kA_{55}\left(\psi_x + \frac{\partial w_0}{\partial x}\right) - kA_{45}\left(\psi_y + \frac{\partial w_0}{\partial y}\right) = R\ddot{u}_0 + I\ddot{\psi}_x$$

$$B_{16}\frac{\partial^2 u_0}{\partial x^2} + (B_{12}+B_{66})\frac{\partial^2 u_0}{\partial x \partial y} + B_{26}\frac{\partial^2 u_0}{\partial y^2} + B_{66}\frac{\partial^2 v_0}{\partial x^2} + 2B_{26}\frac{\partial^2 v_0}{\partial x \partial y} + B_{22}\frac{\partial^2 v_0}{\partial y^2}$$

$$+D_{16}\frac{\partial^2 \psi_x}{\partial x^2} + (D_{12}+D_{66})\frac{\partial^2 \psi_x}{\partial x \partial y} + D_{26}\frac{\partial^2 \psi_x}{\partial y^2} + D_{66}\frac{\partial^2 \psi_y}{\partial x^2} + 2D_{26}\frac{\partial^2 \psi_y}{\partial x \partial y}$$

$$+D_{22}\frac{\partial^2 \psi_y}{\partial y^2} - kA_{45}\left(\psi_x + \frac{\partial w_o}{\partial x}\right) - kA_{44}\left(\psi_y + \frac{\partial w_o}{\partial y}\right) = R\ddot{v}_o + I\ddot{\psi}_x$$

Boundary Conditions

For simplicity, consider the edge parallel to the y-axis. Various boundary conditions associated with the Mindlin plate theory are given as follows.

∘ Simply-supported-movable in the plane of the plate

$$N_x = N_{xy} = w_0 = M_x = M_{xy} = 0 \tag{5.106}$$

○ Hinged-immovable in the plane of the plate

$$u_0 = v_0 = w_0 = M_x = M_{xy} = 0 \tag{5.107}$$

○ Hinged-free in the normal direction

$$N_x = v_0 = w_0 = M_x = M_{xy} = 0 \tag{5.108}$$

○ Hinged-free in the tangential direction

$$u_0 = N_{xy} = w_0 = M_x = M_{xy} = 0 \tag{5.109}$$

○ Clamped

$$u_0 = v_0 = w_0 = \psi_x = \psi_y = 0 \tag{5.110}$$

○ Free

$$N_x = N_{xy} = M_x = M_{xy} = Q_x = 0 \tag{5.111}$$

Example 5.6 *Cylindrical Bending*

Consider a symmetric cross ply laminate $[0/90]_s$ with a span L in the x-direction and infinite in the y-direction. The static transverse loading q is a function of x only, and the edges are uniformly supported in the y-direction. In addition, a state of plane strain is assumed so that $v_0 = \psi_y = 0$. Thus, the plate displacements and rotations must be functions of x only, and the deflected surface is cylindrical. For the symmetric cross ply laminate, we have

$$B_{ij} = 0 \quad , \qquad A_{16} = A_{26} = A_{45} = D_{16} = D_{26} = 0 \tag{a}$$

Thus, the in-plane displacements u_o and v_o are uncoupled from w_o, ψ_x and ψ_y. If the edges ($x = 0,L$) are simply supported, it can be shown that $u_0 = 0$. The equilibrium equations (5.105) then reduce to

$$kA_{55}\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w_o}{\partial x^2}\right) + q = 0 \tag{b}$$

$$D_{11}\frac{\partial^2 \psi_x}{\partial x^2} - kA_{55}\left(\psi_x + \frac{\partial w_o}{\partial x}\right) = 0 \tag{c}$$

The boundary conditions are

$$w_0 = M_x = 0 \quad \text{at} \quad x = 0,L \tag{d}$$

Consider the transverse loading of the form

$$q = q_0 \sin\frac{\pi x}{L} \tag{e}$$

Assume the following solutions

$$w_0 = A \sin \frac{\pi x}{L} \tag{f}$$

$$\psi_x = B \cos \frac{\pi x}{L} \tag{g}$$

which satisfy the boundary conditions (d). Substituting (f) and (g) into (b) and (c) yields

$$\begin{aligned} (kA_{55}\frac{\pi^2}{L^2})A + (kA_{55}\frac{\pi}{L})B &= q_0 \\ (kA_{55}\frac{\pi}{L})A + (D_{11}\frac{\pi^2}{L^2} + kA_{55})B &= 0 \end{aligned} \tag{h}$$

Solving the above equations for A and B, we have

$$A = \frac{L^4 q_0}{\pi^4 D_{11}} + \frac{L^2 q_0}{\pi^2 k A_{55}}$$

$$B = -\frac{L^3 q_0}{\pi^3 D_{11}} \tag{i}$$

The maximum deflection occurs at $x = L/2$ and is given by

$$(w_0)_{max} = (w_0')_{max} + \frac{q_0 L^2}{\pi^2 k A_{55}} \tag{j}$$

where

$$(w_0')_{max} = \frac{L^4 q_0}{\pi^4 D_{11}}$$

is the maximum deflection according to classical plate theory. The second term on the right hand side of (j) is the additional deflection due to transverse shear deformation.

Consider the ratio

$$\frac{(w_o)_{max}}{(w_o')_{max}} = 1 + \frac{\pi^2 D_{11}}{L^2 k A_{55}} \tag{k}$$

For the $[0/90]_s$ laminate, it can easily be shown that

$$A_{55} = \frac{h}{2}(G_{13} + G_{23}) \quad , \quad D_{11} = \frac{h^3}{96}(7Q_{11} + Q_{22})$$

Thus, (k) can be written as

$$\frac{(w_0)_{max}}{(w_0')_{max}} = 1 + \frac{\pi^2 (h/L)^2 (7Q_{11} + Q_{22})}{48k(G_{13} + G_{23})} \tag{l}$$

in which

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \quad , \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \quad \text{and} \quad E_1\nu_{21} = E_2\nu_{12}$$

Note that as $w_0 \longrightarrow w_0'$ as $h/L \longrightarrow 0$.

In the numerical example, three material systems with different degrees of anisotropy are considered. The value of k is taken as $5/6$.

material system 1:

$$\frac{E_1}{E_2} = 40 \quad , \quad \frac{G_{13}}{E_2} = 0.6 \quad , \quad \frac{G_{23}}{E_2} = 0.4 \quad , \quad \nu_{12} = 0.25$$

material system 2:

$$\frac{E_1}{E_2} = 15 \quad , \quad \frac{G_{13}}{E_2} = 0.6 \quad , \quad \frac{G_{23}}{E_2} = 0.4 \quad , \quad \nu_{12} = 0.25$$

material system 3 (isotropic material):

$$\frac{E_1}{E_2} = 1 \quad , \quad \frac{G_{13}}{E_2} = 0.4 \quad , \quad \frac{G_{23}}{E_2} = 0.4 \quad , \quad \nu_{12} = 0.25$$

Figure 5.11 presents the ratio $(w_0)_{max}/(w_0')_{max}$ as a function of h/L for the three material systems. It is evident that the effect of transverse shear deformation on deflection is enhanced by a greater ratio between the in-plane stiffness and the transverse shear stiffness of the composite.

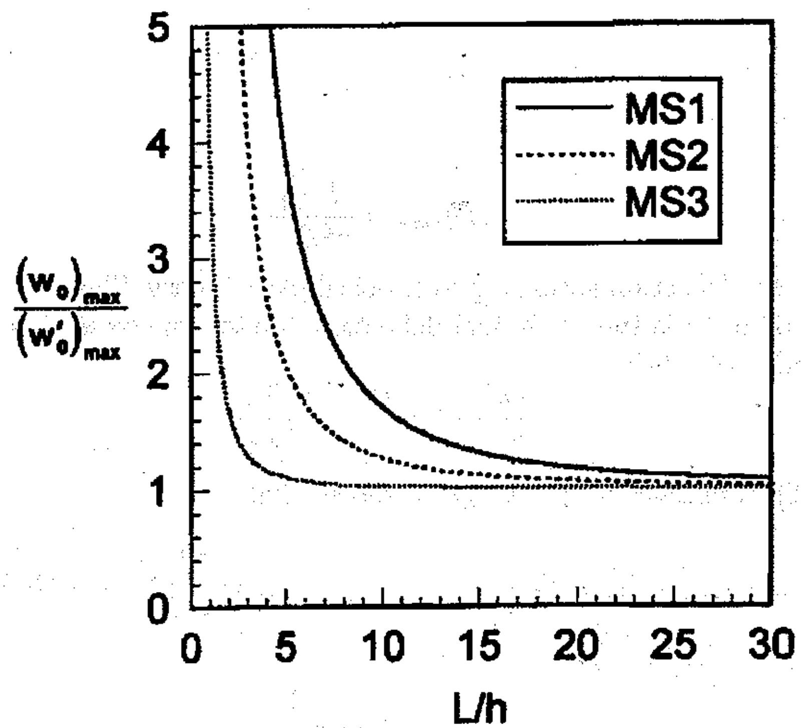


Figure 5.11 Maximum deflection of $[0/90]_S$ laminates of three material systems

under cylindrical bending

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Problems

5.1 Compute the A, B, D matrices for $[0_2/90_2]$, $[90_2/0_2]$, $[0/90/0/90]$, and $[0/90]_s$ laminates. Summarize the characteristics of these matrices. The moduli of the composite are

$E_1 = 140GPa$, $E_2 = 10GPa$, $G_{12} = 7GPa$, $\nu_{12} = 0.3$, ply thickness $= 0.127mm$

5.2 Consider the $[0_2/90_2]$, $[90_2/0_2]$, and $[0/90]_s$ laminates in Prob. 5.1. Find the inplane strains and curvatures produced by uniaxial loading $N_x = 5000N/m$. Compare the deformed shapes of $[0_2/90_2]$ and $[90_2/0_2]$ laminates. If the laminate in-plane stiffness in the $x-$direction is defined as

$$K_x = \frac{N_x}{\varepsilon_x^0}$$

which laminate among the three has the highest in-plane stiffness?

5.3 Find the bending moments that are needed to suppress out-of-plane deflection in the $[0_2/90_2]$ laminate (see Prob. 5.1 for composite properties) subjected to a uniaxial load $N_x = 5000N/m$ and $N_y = N_{xy} = 0$.

5.4 Consider a $[\pm 45]_s$ laminate. If the constituent composite material is highly anisotropic, i.e.,

$$E_1 \gg E_2 \text{ and } E_1 \gg G_{12}$$

show that the effective engineering moduli for the laminate can be expressed approximately as

$$\begin{aligned} E_x &\simeq 4Q_{66} \simeq 4G_{12} \\ G_{xy} &\simeq \frac{Q_{11}}{4} \simeq \frac{E_1}{4} \\ \nu_{xy} &\simeq \frac{Q_{11} - 4Q_{66}}{Q_{11} + 4Q_{66}} \simeq \frac{E_1 - 4G_{12}}{E_1 + 4G_{12}} \end{aligned}$$

Compare these approximate values with the exact values for $AS4/3501$-6 graphite/epoxy composite.

5.5 Derive approximate expressions for the effective engineering moduli for the laminates $[0/90]_s$ and $[0/90/\pm 45]_s$.

5.6 Compare the in-plane longitudinal stiffnesses in the $x-$direction for $[\pm 30/0]_s$ and $[30_2/0]_s$ laminates of $AS4/3501$-6 graphite/epoxy composite. Which is stiffer?

5.7 Plot the effective moduli E_x, G_{xy}, and ν_{xy} versus θ for the angle-ply laminate $[\pm\theta]_s$ of $AS4/3501$-6 graphite/epoxy composite.

5.8 Find the shear strains (γ_{xy}) in the $AS4/3501-6$ graphite/epoxy composite $[\pm 45]_s$ and $[0/90]_s$ laminates subjected to the shear loading $N_{xy} = 1000N/m$. Also find the lamina stresses σ_{11}, σ_{22}, and σ_{12}. If the maximum shear strength of the composite is $|\sigma_{12}| = 100MPa$, what are the shear loads (N_{xy}) the two laminates can carry?

5.9 Use $\Delta T = -150°C$ to estimate the curing stresses in the $[0/90]_s$ laminates of the following two composite materials.

AS4/3501-6 graphite/epoxy:	$E_1 = 140GPa$, $E_2 = 10GPa$, $G_{12} = 7GPa$ $\nu_{12} = 0.3$, $\alpha_1 = -1 \times 10^{-6}/°C$, $\alpha_2 = 26 \times 10^{-6}/°C$
S glass/epoxy:	$E_1 = 45GPa$, $E_2 = 9GPa$, $G_{12} = 4.5GPa$ $\nu_{12} = 0.3$, $\alpha_1 = 5 \times 10^{-6}/°C$, $\alpha_2 = 26 \times 10^{-6}/°C$

5.10 A [0/90/0] laminate of $AS4/3501$-6 graphite/epoxy composite is confined in the 0° direction. (i.e., $\varepsilon_x^0 = 0$) but free to expand (or contract) in the 90° direction (i.e., $N_y = 0$). Find the lamina stresses for $\Delta T = -100°C$.

5.11 A $[0/90]_s$ laminate of AS4/3501-6 composite is fixed on all sides. Find the reaction forces N_x and N_y and the lamina stresses for $\Delta T = -100°C$.

5.12 Do Prob. 5.11 for the $[\pm 30]_S$ laminate. Also find the effective thermal expansion coefficients $\bar{\alpha}_x$ and $\bar{\alpha}_y$.

5.13 Due to the presence of curing stresses, the [0/90] laminate would warp after curing. If the laminate is constrained so the $\kappa_y = \kappa_{xy} = 0$ but $\kappa_x \neq 0$, find the curvature κ_x after curing. Assume $\Delta T = -150$°C. The composite is AS4/3501-6.

5.14 Find the mechanical stresses σ_{xx}, σ_{yy}, and σ_{xy} that are needed to cancel the thermal strains induced by a temperature rise ΔT in [0] and [45] composite panels.

5.15 A [0/90] unsymmetric crossply laminate is subjected to the in-plane loads $N_x = N$ (see Fig. 5.12). The plate is simply supported along $x = \pm a$. Assume that the laminate is very long in the $y-$direction (so that $v_0 = 0$, $\kappa_y = 0$). Find the transverse deflection $w_0(x)$ in terms of A, B, D matrices.

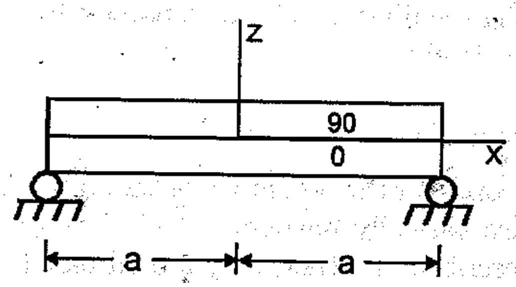
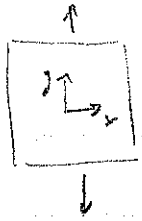


Figure 5.12

Fracture mechanics



$\sigma_{xx} = \frac{K}{\sqrt{2\pi r}} + \dots$

$K_I \geq K_c$ (fracture)

homogeneous, orthotropic solid.
(testing of fiber direction, transverse direction, etc
- Basic strengths obtained by testing
- Tension: Tensile strength (X)
- Compression: (X')
- Simple (tension/compression) transverse direction (Y, Y')
- Shear: (S_{12}) $x_1 - x_2$

  S_{13} $x_1 - x_3$

Start with Basic element: lamina

Micromechanics: fiber + matrix
composite elastic constants.

Chapter 6

STRENGTH CRITERIA FOR COMPOSITES

As discussed in the previous chapters, the anisotropic property of composite materials leads to different stiffnesses in different loading directions. In the same manner, the ultimate load a composite can carry also depends on the fiber orientation. Unlike the elastic moduli for which the values at different orientations can be obtained from the coordinate transformation law for elastic constants, no such transformation law exists for strengths of composites.

The strength theories to be discussed in this chapter essentially follow the path of continuum plasticity. The advantages of such theories are their simplicity and the use of elasticity results. The approach is particularly useful to structural designers who must know the load-carrying capacity of a composite material subjected to a complex state of stress. All these theories try to predict the strength of a unidirectional lamina or a laminate based upon the three basic strengths X, Y, S of a lamina loaded along the fiber direction, transverse to the fiber-direction, and under in-plane shear, respectively.

6.1 Failure Criteria for a Lamina (are empirical)

There exist a few theories for the prediction of the strength of unidirectional composite materials. These are basically phenomenological theories in which detailed failure processes are not described. Further, they are all based on linear elastic analysis.

Maximum Stress Criterion (no stress interaction:

Let X be the maximum tensile stress that the lamina can take in the x—direction (fiber direction), Y be that in the x_2-direction (transverse to the fiber), and S be the in-plane shear strength. Since the compressive strengths in composite materials are usually different from the tensile strengths, X' and Y' will be used to denote the compressive strengths.

Maximum stress theory states that a lamina fails if a state of stress (σ_{11}, σ_{22}, σ_{12}) is produced by an applied load such that

$$\sigma_{11} \geq X \tag{6.1}$$

(tensile stress on fiber direction)

or

$$\sigma_{22} \geq Y \tag{6.2}$$

or

$$|\sigma_{12}| \geq S \tag{6.3}$$

If the normal stresses are compressive, then the conditions given by (6.1) and (6.2) are replaced by

$$\sigma_{11} \leq X' \tag{6.4}$$

$$\sigma_{22} \leq Y' \tag{6.5}$$

respectively. Note that X' and Y' are negative values.

The maximum stress theory as given by (6.1-6.3) implies that three independent failure modes are assumed and each mode is governed by a single stress component.

Maximum Strain Criterion

If, instead of the maximum stresses, the maximum strains are registered at the points of failure of a lamina, a criterion similar to the maximum stress theory can be established. Specifically, a state of deformation would cause failure of a lamina if any of the following inequalities is satisfied:

$$\epsilon_{11} \geq X_\epsilon \quad \text{tension.} \tag{6.6}$$

$$\epsilon_{22} \geq Y_\epsilon \tag{6.7}$$

$$|\gamma_{12}| \geq S_\epsilon \tag{6.8}$$

when X_ϵ, Y_ϵ , and S_ϵ are the ultimate tensile strains in the x_1- and x_2-directions, and the maximum shear strain in the x_1-x_2 plane, respectively. Again, to account for different compressive strengths, (6.6) and (6.7) should be replaced by

$$\epsilon_{11} \leq X'_\epsilon \quad \text{compression.} \tag{6.9}$$

and

$$\epsilon_{22} \leq Y'_\epsilon \tag{6.10}$$

respectively, where X'_ϵ and $Y'\epsilon$ are the ultimate compressive strains (negative values). If a composite material behaves linearly elastic up to failure as shown in Fig. 6.1, then

if linear : $$X_\epsilon = \frac{X}{E_1},\ Y_\epsilon = \frac{Y}{E_2},\ S_\epsilon = \frac{S}{G_{12}}$$

according to ε_{xy} criterion.
$\varepsilon_{22} = 0 \le \gamma_{12}$

$$X'_\epsilon = \frac{X'}{E_1},\ Y'_\epsilon = \frac{Y'}{E_2} \tag{6.11}$$

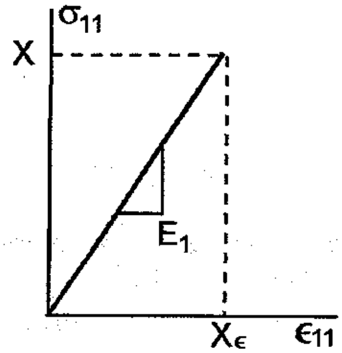
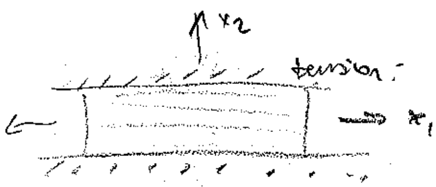


Fig. 6.1 Longitudinal strength in linear composite

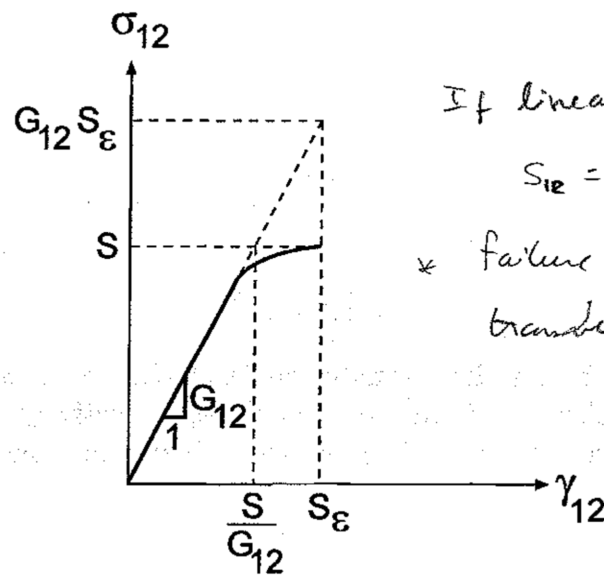
However, actual stress-strain curves for a composite usually exhibit some nonlinearity, especially the in-plane shear deformation as shown in Fig. 6.2. It is obvious that

$$X_\epsilon \simeq \frac{X}{E_1},\ Y_\epsilon > \frac{Y}{E_2},\ S_\epsilon > \frac{S}{G_{12}}$$

assume linear elastic materials.



$\varepsilon_{11} \neq 0$
$\varepsilon_{22} \neq 0$
$\sigma_{22} = 0$



If linear material.
$S_{12} = G_{12} \cdot \gamma_\varepsilon$
* failure is assumed in transverse direction.

Fig. 6.2 Nonlinear stress and strain relation in shear

Thus, using the measured strength S and that obtained from the measured strain S_ϵ, namely $G_{12}S_\epsilon$, the maximum stress criterion will predict different results. In general, the maximum stress and the maximum strain criteria yield different results for strength predictions even if linear stress-strain relations are imposed and the relations of (6.11) are assumed true.

One of the obvious weaknesses of the maximum stress and maximum strain criteria is the fact that they disregard the combined effects of stress on failure. Such practice may lead to unconservative strength predictions when multi-axial stresses or strains are present. The failure criteria discussed in the following provide the remedy for this deficiency.

3) Hill-Tsai Criterion (Stress Interaction). Yield criteria for orthotropic materials

For isotropic materials, a popular yield criterion is the von Mises yield criterion which states that yielding of a material begins when the distortion energy density equals the distortion energy density at yield in simple tension. The distortion energy is the total strain energy density less the dilational strain energy. Inherent in the theory is the assumption that yielding is due solely to shear deformation and that pure dilation (a volume change) such as deformation due to a hydrostatic pressure would not produce yielding.

If $\sigma_{11} < 0$, $X \to X'$; $\sigma_{22} < 0$, $Y \to Y'$

Hill [6.1] extended von Mises' isotropic yield criterion to account for orthotropy of some materials. He obtained a yield criterion in the form

6 unknowns constants.

if $X = |X'|$, $Y = |Y'|$ no change

$$\boxed{F(\sigma_{22}-\sigma_{33})^2+G(\sigma_{33}-\sigma_{11})^2+H(\sigma_{11}-\sigma_{22})^2+2L\sigma_{23}^2+2M\sigma_{31}^2+2N\sigma_{12}^2=1} \quad (6.12) \quad *1$$

The coefficients F, G, H, L, M, and N are to be determined from the yielding behavior.

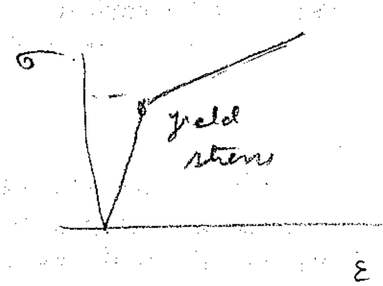
Tsai [6.2] used the expression (6.12) as a failure criterion for unidirectional composites by reinterpreting yield stresses as failure stresses. In so doing, the coefficients F, G, H, L, M, and N are determined from tests of the longitudinal strengths in the three material principal directions and the shear strengths in the three orthogonal planes of symmetry. Six equations result, i.e.,

$$2F = \frac{1}{Y^2}+\frac{1}{Z^2}-\frac{1}{X^2}$$

$$2G = \frac{1}{Z^2}+\frac{1}{X^2}-\frac{1}{Y^2}$$

$$2H = \frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2} \qquad 2M = \frac{1}{S_{31}^2} \qquad (6.13)$$

$$2L = \frac{1}{S_{23}^2} \qquad 2H = \frac{1}{S_{12}^2}$$



Mises yield criteria:

Distortion energy equals a critical value of material yield for isotropic materials

Assumed:

- Transverse isotropic
- $\sigma_{33} = \sigma_{13} = \sigma_{32} = 0$
- $Y = Z$

given a state of plane stress: σ_{11}, σ_{22}, σ_{12} are found.
want to see if they produce failure: plug in numbers in eq: (*)
$f(\sigma_{ij}) \geq 1$: what type of failure?: if σ_{11} reaches max strength stress first,
there is failure in (1) direction.

To find mode of failure:
compare ratios

$\frac{\sigma_{11}}{X}$, $\frac{\sigma_{22}}{Y}$, $\frac{\sigma_{12}}{S}$ and see

which one is the largest: largest ratio determines mode of failure. (guess)

$$2M = \frac{1}{S_{31}^2}$$

$$2N = \frac{1}{S_{12}^2}$$

in which X, Y, Z are the uniaxial tensile strengths in the material's principal directions x_1, x_2, and x_3, respectively, and S_{23}, S_{31}, S_{12} are the shear strengths in the x_2-x_3, x_3 - x_1, and x_1 - x_2 planes, respectively.

For a lamina of unidirectionally fiber-reinforced composite in a state of plane stress parallel to the x_1 - x_2 plane, the yield criterion given by (6.12) reduces to

$$(G+H)\sigma_{11}^2 - 2H\sigma_{11}\sigma_{22} + (F+H)\sigma_{22}^2 + 2N\sigma_{12}^2 = 1 \tag{6.14}$$

If, in addition, we assume that the composite is transversely isotropic, i.e., $Y = Z$, then the criterion is further simplified to

dominant

$$(*)\quad \boxed{\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 - \left(\frac{\sigma_{11}}{X}\right)\left(\frac{\sigma_{22}}{X}\right) + \left(\frac{\sigma_{12}}{S}\right)^2 = 1} \tag{6.15}$$

if < 1 no failure

where S = S_{12}. This is the so-called *Hill-Tsai failure criterion.*

In Hill's original yield criterion, tensile and compressive yield properties were assumed the same. For fiber composites, the tensile strength may be significantly different from the compressive strength. The strength criterion given by (6.15) can be used in the four quadrants formed by the normal stresses with the proper strengths values. For example, in the third quadrant where σ_{11} and σ_{22} are both negative, X' and Y' are to be used in place of X and Y, respectively, in the failure criterion.

Tsai-Wu Criterion

A general stress failure criterion can be expressed in the form

$$\boxed{f(\sigma_i) = 1}, i = 1, 2, ..., 6 \tag{6.16}$$

$\sigma_{11} = \sigma_1$
$\sigma_{22} = \sigma_2$
$\sigma_{33} = \sigma_3$
$\sigma_{23} = \sigma_4$
$\sigma_{13} = \sigma_5$
$\sigma_{12} = \sigma_6$

Since strength is an inherent material property, the failure criterion must be invariant with respect to the choice of coordinate systems. An explicit form which satisfies this requirement was proposed by Tsai and Wu [6.3] as

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j + F_{ijk}\sigma_i\sigma_j\sigma_k + ... = 1 \tag{6.17}$$

(Taylor series expansion)

where F_i, F_{ij}, ..., etc., are material parameters. In (6.17) the summation convention over repeated indices is used.

To ensure the said invariant property, each term in (6.17) must be a scalar. When the stress components σ_i are viewed as a second order tensor, then according

$f(\sigma_i) =$

$F_1\sigma_1 + F_2\sigma_2 + \ldots$
$+ F_{11}\sigma_1\sigma_1 + \ldots F_6\sigma_6\sigma_6$

to the quotient rule, the six independent components F_i can also be expanded to form a second order tensor. By the same token, F_{ij} can be expanded to form a 4th order tensor. The tensorial properties of the expanded forms F_i, F_{ij} ... provide the coordinate transformation laws for the material constants F_i, F_{ij} ..., .

For orthotropic materials, it is more convenient to work with the coordinates which are parallel to the material principal directions, x_1, x_2, and x_3. With respect to this particular coordinate system, extension and shear are uncoupled. By using the physical argument that strength should not depend on the sign of the shear stresses, σ_4, σ_5, and σ_6, the following conditions are obtained from (6.17),

$$F_4 = F_5 = F_6 = 0 \tag{6.18}$$

in linear stress terms, and

$$\begin{aligned} F_{14} &= F_{15} = F_{16} = 0 \\ F_{24} &= F_{25} = F_{26} = 0 \\ F_{34} &= F_{35} = F_{36} = 0 \\ F_{45} &= F_{46} = F_{56} = 0 \end{aligned} \tag{6.19}$$

in quadratic terms. Similar conditions on the higher order coefficients can be obtained.

If we retain the linear and quadratic terms in (6.17), the failure criterion is given explicitly as

$$\begin{aligned} &F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{33}\sigma_3^2 + 2F_{12}\sigma_1\sigma_2 \\ &+2F_{23}\sigma_2\sigma_3 + 2F_{13}\sigma_1\sigma_3 + F_{44}\sigma_4^2 + F_{55}\sigma_5^2 + F_{66}\sigma_6^2 = 1 \end{aligned} \tag{6.20}$$

If a state of plane stress parallel to the x_1 - x_2 plane exists, i.e., $\sigma_3 = \sigma_4 = \sigma_5 = 0$, then the failure criterion further reduces to

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + 2F_{12}\sigma_1\sigma_2 + F_{66}\sigma_6^2 = 1 \tag{6.21}$$

Thus, six material constants are to be determined in order to complete the failure criterion . This task can be carried out in part by performing simple tension, compression, and shear tests up to the failure point. Using these test results and (6.21), we obtain

$$F_1 = \frac{1}{X} + \frac{1}{X'}, \; F_2 = \frac{1}{Y} + \frac{1}{Y'}$$

$$F_{11} = \frac{-1}{XX'}, \; F_{22} = \frac{-1}{YY'}, \; F_{66} = \frac{1}{S_{12}} \tag{6.22}$$

The remaining constant F_{12} can be determined only from a state of stress in which both σ_1 and σ_2 are present. Such a state can easily be produced by an off-axis test. Let σ_u be the ultimate stress for a unidirectional composite in plane stress under off-axis loading. The stresses referred to the material principal axes are

$F_{14} = F_{15} = F_{16} = 0$
$F_{24} = 0$
$F_{34} = 0$
$F_{45} = 0$
If assume plane-stress
$F_1\sigma_1 = F_2\sigma_2 + F_1\sigma_1^2 + F_{32}\sigma_2$

$$
\begin{aligned}
\sigma_{11} &= \sigma_1 = \frac{1}{2}\sigma_u(l + cos2\theta) \\
\sigma_{22} &= \sigma_2 = \frac{1}{2}\sigma_u(l - cos2\theta) \\
\sigma_{12} &= \sigma_6 = -\frac{1}{2}\sigma_u sin2\theta)
\end{aligned}
\tag{6.23}
$$

Substituting (6.23) into (6.21), we obtain

$$
\begin{aligned}
2F_{12} = & [4 - 2F_1\sigma_u(l + cos2\theta) - 2F_2\sigma_u(l - cos2\theta) \\
& - F_{11}\sigma_u^2(1 + cos2\theta)^2 - F_{22}\sigma_u^2(1 - cos2\theta)^2 \\
& - F_{66}\sigma_u^2 \sin^2 2\theta]/\sigma_u^2(1 - \cos^2 2\theta)
\end{aligned}
\tag{6.24}
$$

At first glance the expression given by (6.24) suggests that F_{12} depends on the angle θ. Since F_{12} is presumably a material constant and should not vary with θ, its invarience is expected to be achieved by the compensating effect of the $\theta-$dependent σ_u. It has been found that the failure criterion is not sensitive to the value of F_{12}, and it is suggested to set

$$F_{12} = \frac{1}{2XX'}$$

Hence, the tensor polynomial criterion given by (6.21) reduces to the *Hill-Tsai criterion* if $X = -X'$ is assumed.

If, as in Hill's yield criterion, the tensile strengths are assumed to be the same as the compressive strengths, i.e.,

$$X = -X' \, , Y = -Y' \, , Z = -Z' \tag{6.25}$$

then the linear terms in (6.20) drop out. The Hill's yield criterion can be deduced from this resulting equation by further requiring that the criterion is independent of hydrostatic pressure $\sigma_p = 1/3(\sigma_1 + \sigma_2 + \sigma_3)$. This condition states that

$$\frac{\partial}{\partial \sigma_p}(F_{ij}\sigma_i\sigma_j) = 0 \tag{6.26}$$

or, by using the chain rule,

$$\frac{\partial}{\partial \sigma_1}(F_{ij}\sigma_i\sigma_j)\frac{\partial \sigma_1}{\partial \sigma_p} + \frac{\partial}{\partial \sigma_2}(F_{ij}\sigma_i\sigma_j)\frac{\partial \sigma_2}{\partial \sigma_p} + \frac{\partial}{\partial \sigma_3}(F_{ij}\sigma_i\sigma_j)\frac{\partial \sigma_3}{\partial \sigma_p} = 0 \tag{6.27}$$

$F_4 = F_5$, $F_6 = 0$

The result is

$$(F_1, +F_{12} + F_{13})\sigma_1 + (F_{12} + F_{22} + F_{23})\sigma_2 + (F_{13} + F_{23} + F_{33})\sigma_3 = 0 \tag{6.28}$$

Since this condition must hold for arbitrary stresses, it requires that

$$\begin{aligned} F_{11} + F_{12} + F_{13} &= 0 \\ F_{12} + F_{22} + F_{23} &= 0 \\ F_{13} + F_{23} + F_{33} &= 0 \end{aligned} \tag{6.29}$$

From these equations, the coefficients associated with the cross-product terms can be expressed as

$$\begin{aligned} 2F_{12} &= F_{33} - F_{11} - F_{22} \\ 2F_{23} &= F_{11} - F_{22} - F_{33} \\ 2F_{13} &= F_{22} - F_{11} - F_{33} \end{aligned} \tag{6.30}$$

With conditions given by (6.25) and (6.29), it is quite straightforward to show that

coefficients dice 6.
X, X', Y, Y', S
$\Rightarrow F_{11} = \frac{1}{X} + \frac{1}{X'}$
$F_{11}\ \frac{1}{XX'} - F_2$

$$\begin{aligned} F_{11} &= \frac{1}{X^2}, F_{22} = \frac{1}{Y^2}, F_{33} = \frac{1}{Z^2} \qquad F_{12} = \frac{1}{2XX'} + \\ F_{44} &= \frac{1}{S_{23}^2}, F_{55} = \frac{1}{S_{13}^2}, F_{33} = \frac{1}{S_{12}^2} \qquad F_{66} = \end{aligned} \tag{6.31}$$

and that *Tsai-Wu criterion* reduces to Hill's yield criterion for orthotropic materials.

Separate Mode Criterion (Separate failure)

There are two major modes of failure in unidirectional fiber composites, i.e., fiber breakage and matrix cracking. The failure condition for each mode is governed by the individual state of stress in the fiber and matrix, respectively. If *Hill-Tsai* or *Tsai-Wu failure criterion* is used, these two modes cannot be distinguished. Usually, the largest ratio among σ_{11}/X, σ_{22}/Y, and σ_{22}/S is used to identify the associated failure mode.

Consider failures of the fiber and matrix separately. Using a quadratic form in stress, similar to the Hill's criterion, as the failure criterion for the fiber, we have

Fiber failure mode

$\frac{\sigma_{11}}{X} \geq 1$ for tension

$$\boxed{\underbrace{\left(\frac{\sigma_{11}^f}{X_f}\right)^2}_{\text{dominant}} - \left(\frac{\sigma_{11}^f}{X_f}\right)^2\left(\frac{\sigma_{22}^f}{X_f}\right)^2 + \left(\frac{\sigma_{22}^f}{Y_f}\right)^2 + \left(\frac{\sigma_{12}^f}{S_f}\right)^2 = 1} \tag{6.32}$$

when σ_{11}^f are the stresses in the fiber, and X_f, Y_f, and S_f are the major strengths in the fiber. From the simple micromechanical model discussed in Section 3.3, it is seen that σ_{22}^f and σ_{12}^f are approximately the same as the corresponding stresses in the

$\frac{\sigma_{11}}{X'} = 1$ compression.

matrix. Thus, σ_{22}^f and σ_{12}^f cannot be raised to a significant level as compared with Y_f and S_f without causing matrix failure. In view of this, (6.32) can be simplified to

$$\left(\frac{\sigma_{11}^f}{X_f}\right)^2 = 1 \tag{6.33}$$

Since σ_{11}^f is proportional to the composite stress σ_{11}, (6.33) suggests the suitable failure criterion for fiber breakage as

$$\left(\frac{\sigma_{11}}{X}\right)^2 = 1 \tag{6.34}$$

fiber failure

or

$$\frac{\sigma_{11}}{X} = 1 \text{ for tension}$$

and

$$\frac{\sigma_{11}}{X'} = 1 \text{ for compression}$$

where X and X' are the longitudinal tensile and compressive strengths of the composite, respectively.

matrix Failure

A quadratic polynomial in stress similar to (6.32) can be assumed to predict failure of the matrix. If we assume that σ_{11}^m is small (since the fiber takes the load in proportion to its Young's modulus), then the failure criterion reduces to

$$\left(\frac{\sigma_{22}^m}{Y_m}\right)^2 + \left(\frac{\sigma_{12}^m}{S_m}\right)^2 = 1 \tag{6.35}$$

This leads to the criterion for matrix failure in the unidirectional composite as

$$\left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\sigma_{12}}{S}\right)^2 = 1 \tag{6.36}$$

Matrix Failure

In this form, the transverse strength Y and the shear strength S of the composite may include the interfacial failure between the fiber and matrix.

The failure criterion given by (6.34) and (6.36) was proposed by Hashin and Rotem [6.4].

6.2 Analysis of Lamina Strength

When loading is uniaxial and is applied along one of the material principal axes, the use of the failure criteria discussed in the previous section becomes trivial. We now consider a panel of unidirectional fiber composite under off-axis loadings as shown in Fig. 6.3. The state of stress in the panel is

$$\sigma_{xx} = \sigma_o, \sigma_{yy} = \sigma_{xy} = 0$$

where σ_0 can be either positive (tensile) or negative (compressive). Corresponding to this state of stress, the stress components referred to the x_1 - x_2 system are obtained by coordinate transformation as

$$\begin{aligned} \sigma_{11} &= \sigma_o cos^2\theta \\ \sigma_{22} &= \sigma_o sin^2\theta \\ \sigma_{12} &= -\sigma_o sin\theta cos\,\theta \end{aligned} \tag{6.37}$$

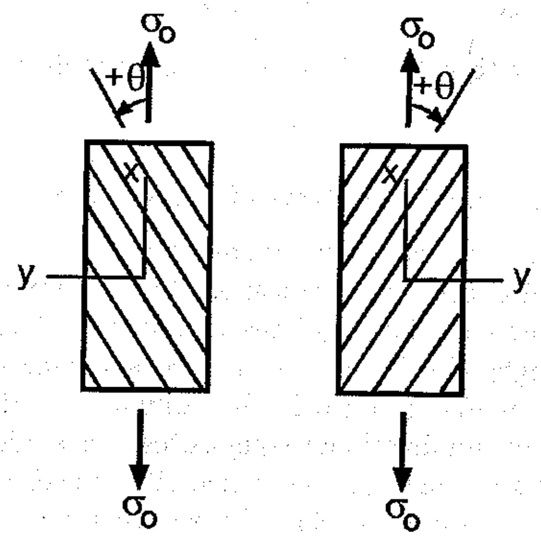


Fig. 6.3 Coordinate system for off-axis composite panel

The corresponding strains are

$$\begin{aligned} \epsilon_{11} &= \frac{\sigma_o}{E_1}\left(\cos^2\theta - \nu_{12}\sin^2\theta\right) \\ \epsilon_{22} &= \frac{\sigma_o}{E_2}\left(\sin^2\theta - \nu_{21}\cos^2\theta\right) \\ \gamma_{12} &= -\frac{\sigma_o}{G_{12}}\sin\theta\ \cos\theta \end{aligned} \tag{6.38}$$

The uniaxial strength is the smallest value among the following three

$$\begin{aligned} \sigma_o &= X/cos^2\theta \\ \sigma_o &= Y/sin^2\theta \\ \sigma_o &= S/sin\theta cos\theta \end{aligned} \tag{6.39}$$

if the *maximum stress criterion* is used.

If the *maximum stress criterion* is used, then the uniaxial strength is determined by the smallest of the following

$$\begin{aligned} \sigma_o &= X_\epsilon E_1/(cos^2\theta - \nu_{12}sin^2\theta) \\ \sigma_o &= Y_\epsilon E_2/(sin^2\theta - \nu_{21}cos^2\theta) \\ \sigma_o &= S_\epsilon G_{12}/sin\theta cos\theta \end{aligned} \tag{6.40}$$

It should be noted that then the compressive strengths X', Y', and X'_ϵ, Y'_ϵ must be used to replace X, Y, and X_ϵ, Y_ϵ. respectively, if a compression load is applied.

The prediction of the uniaxial strength of the panel using the *Hill-Tsai criterion* can be obtained by substituting the stresses given by (6.37) into (6.15). The maximum uniaxial strength is found to satisfy the following equation

$$\frac{1}{\sigma_o^2} = \frac{1}{X^2}\cos^4\theta + \left(\frac{1}{S^2} - \frac{1}{X^2}\right) cos^2\theta sin^2\theta + \frac{1}{Y^2} sin^4\theta \tag{6.41}$$

In general, these three criteria lead to somewhat different strength predictions.

6.3 Elementary Strength Analysis of Laminates

Laminate failure analysis:

*Find stresses on each lamina:

* failure condition for each lamina

The failure mechanisms and modes in laminated composites are much more complex than those in the unidirectional composite. In the laminate, the laminar failure usually does not imply total failure of the laminate as the rest of the laminae may be able to sustain higher loads. Due to the constraining effect from the adjacent laminae, the in situ lamina strength in the laminate could be substantially higher than that measured in unidirectional composites. In addition, failure modes such as delamination may occur in laminates which often require three-dimensional stress analysis rather than the laminated plate theory discussed in Chapter 5.

The analysis of laminate strength presented in this section is basically a laminar failure analysis in conjunction with laminate stiffness reduction due to laminar failure. The procedure is quite straightforward and, to a certain extent, arbitrary. Two major modes of failure are assumed, i.e., the fiber mode and the matrix mode. The former is signified by fiber breakage; the latter, by matrix cracking. The mode of failure can be identified easily if the maximum stress criterion and the maximum strain criterion are used. For example, when the failure condition is satisfied by $\sigma_{11} \geq X$, then it is a fiber failure mode; otherwise, it would be a matrix failure mode. When the *Hill-Tsai* or *Tsai-Wu criterion* is employed, the mode of failure is identified by comparing the ratios σ_{11}/X, σ_{22}/Y, and σ_{12}/S and assuming that the stress that produces the highest ratio causes the failure.

For a given load, the stresses in each lamina are first calculated and examined with a chosen laminar failure criterion . After a lamina is found to have failed, the laminar properties are modified to reflect the mode of failure. There are several ways to modify the laminar stiffnesses, each of which has a certain degree of arbitrariness . The amount of stiffness reduction is still an open question. The following are several methods commonly used in this type of laminate strength prediction.

Parallel Spring Model

A simplistic model for accounting for laminate stiffness reduction due to progressive ply failure is the parallel spring model as shown in Fig. 6.4. Each spring set represents a lamina which consists of two parallel springs representing the fiber (longitudinal) and matrix (transverse and shear) deformation modes. When fiber breakage mode occurs, E_1 is reduced to zero (or to any designated value). On the other hand, E_2 and G_{12} are reduced if failure is in the matrix.

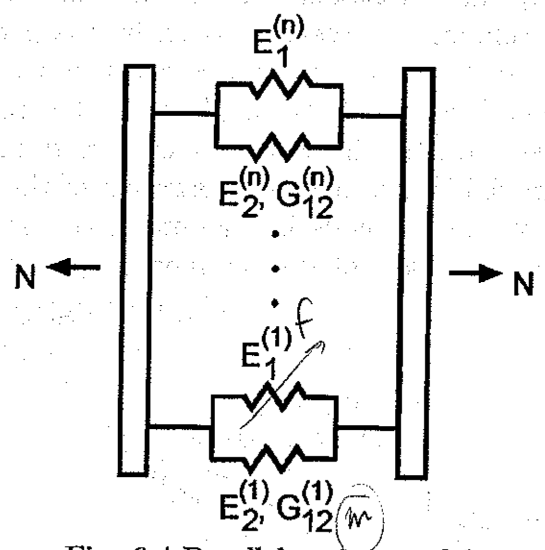


Fig. 6.4 Parallel spring model

Under monotonic loading, this model yields a stress-strain curve as shown in Fig. 6.5 . Since ply failure is assumed to take place suddenly, there is a jump in strain at each stress level corresponding to a ply failure. After a ply failure, the laminate stiffness is reduced and the subsequent loading will follow the path determined by the new laminate stiffness.

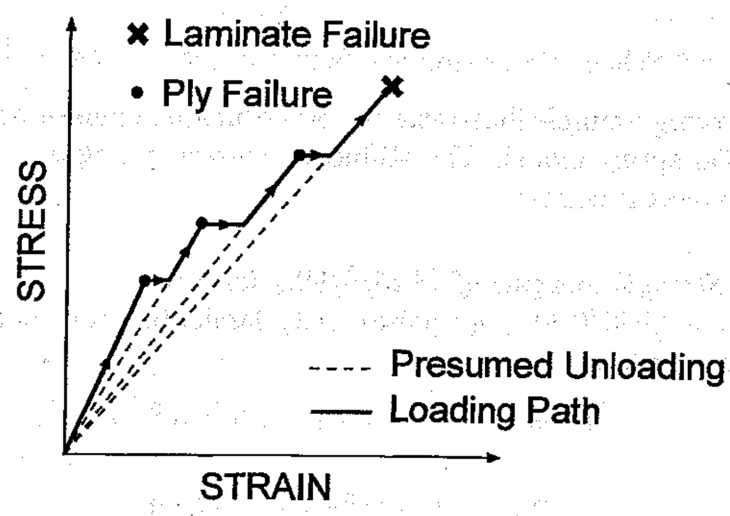


Fig. 6.5 Schematic failure process in laminates according to parallel spring model

It is important to note that, after the first ply failure, the loading path is governed by the current stiffness (see the dashed lines in Fig. 6.5). Thus, between two ply failure events, the laminate stress-strain relation is linear, although the entire progressive failure process may produce a highly nonlinear stress-strain relation. This greatly simplifies the progressive failure analysis in a laminate.

In reality, one seldom observes distinct jumps in strain as illustrated in Fig. 6.5. The post first-ply failure stress-strain curve appears more or less smooth. This does not necessarily rule out the scenario given by the parallel spring model. Experimental results have shown that matrix cracking in a lamina of a laminate is a stochastic process, and it never occurs completely and at the same time as assumed by the strength model. In fact, the matrix crack density increases gradually as the load increases. Figure 6.6 is a schematic illustration of the matrix cracking process associated with a single failure mode in a lamina This staircase type response may appear as a smooth curve in the global scale when the jumps are small and numerous.

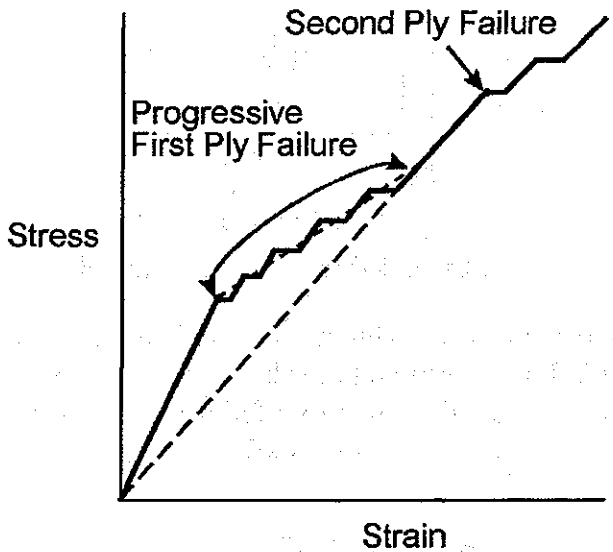


Fig. 6.6 Schematic progressive failure in the ply that fails first

The following example illustrates the procedure in laminate strength prediction using the parallel spring model. The stiffnesses corresponding to a failure mode are reduced to zero once it occurs.

Example 6.1 *Strength analysis of* $[\pm 45/0/90]_S$ *laminate*

Consider a $[\pm 45/0/90]_S$ graphite/epoxy laminate under simple tension N_x. The laminar properties are

$$E_1 = 20 \times 10^6 psi, E_2 = 1.5 \times 10^6 psi$$

$$G_{12} = 1.0 \times 10^6 psi, \nu_{12} = 0.29$$

$$X = 310ksi, Y = 9ksi, S = 15ksi \tag{a}$$

$$X' = -310ksi, Y' = -30ksi$$

$$\text{thickness of lamina} = .005"$$

The stiffness matrices $[\overline{Q}_{ij}]$ are obtained as

$$[\overline{Q}]_{0^\circ} = \begin{bmatrix} 20.13 & 0.44 & 0 \\ 0.44 & 1.51 & 0 \\ 0 & 0 & 1.00 \end{bmatrix} \times 10^6 psi$$

$$[\overline{Q}]_{\pm 45^\circ} = \begin{bmatrix} 6.63 & 4.63 & \pm 4.65 \\ 4.63 & 6.63 & \pm 4.65 \\ \pm 4.65 & \pm 4.65 & 5.19 \end{bmatrix} \times 10^6 psi$$

$$[\overline{Q}]_{90^\circ} = \begin{bmatrix} 1.51 & 0.44 & 0 \\ 0.44 & 20.13 & 0 \\ 0 & 0 & 1.00 \end{bmatrix} \times 10^6 psi$$

The [A] is

$$[A] = \begin{bmatrix} 3.50 & 1.01 & 0 \\ 1.01 & 3.50 & 0 \\ 0 & 0 & 1.24 \end{bmatrix} \times 10^5 \; lb/in$$

The inversion of [A] is

$$[A]^{-1} = \begin{bmatrix} 3.13 & -0.91 & 0 \\ -0.91 & 3.13 & 0 \\ 0 & 0 & 8.08 \end{bmatrix} \times 10^{-6} in/lb$$

For a uniaxial load N_x $(N_y = N_{xy} = 0)$, the laminate strains are

$$\left\{ \begin{array}{c} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{array} \right\} = [A]^{-1} \left\{ \begin{array}{c} N_x \\ 0 \\ 0 \end{array} \right\} = N_x \left\{ \begin{array}{c} 3.13 \\ -0.91 \\ 0 \end{array} \right\} \times 10^{-6}$$

The laminar stresses σ_{xx}, σ_{yy} and σ_{xy} can be computed using

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\}_k = [\overline{Q}]_k \left\{ \begin{array}{c} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{array} \right\}$$

By using the coordinate transformation law for stress, we next obtain the stress components σ_{11}, σ_{22}, and σ_{12} in each lamina . The results are

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{array}\right\}_{0^\circ} = N_x \left\{\begin{array}{c}62.6\\ 0.003\\ 0\end{array}\right\} \tag{b}$$

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{array}\right\}_{\pm 45^\circ} = N_x \left\{\begin{array}{c}22.82\\ 2.18\\ \pm 4.04\end{array}\right\} \tag{c}$$

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{array}\right\}_{90^\circ} = N_x \left\{\begin{array}{c}-16.90\\ 4.33\\ 0\end{array}\right\} \tag{d}$$

To determine the failure load for each lamina, we choose the *Hill-Tsai criterion.* By substituting (b)-(d) into the failure criterion, we can obtain the critical N_{xcr} for failing each lamina For the 90o -ply, we have

$$N_{xcr}^2 \left\{\left(\frac{16.9}{310000}\right)^2 + \left(\frac{4.33}{9000}\right)^2 + \frac{16.9 \times 4.33}{310000^2}\right\} = 1$$

from which

$$N_{xcr} = \underline{2063\ lb/in}$$

By using a similar procedure, thc critical loads for failing other laminas can be calculated. We obtain $N_{xcr} = 49501b/in$ for thc 0° ply and $N_{xcr} = 2711lb/in$ for the $+45^\circ$ -plies . In this case, the critical load for the 90° -ply turns out to be the smallest and, thus, is the critical load for the laminate. This load is the first ply-failure load

To determine the mode of failure, the stress-strength ratios are calculated as

$$\frac{\sigma_{11}}{X'} = \frac{16.9 N_{xcr}}{310000} = 0.11$$

$$\left(\frac{\sigma_{12}}{S}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 = \frac{4.33 N_{xcr}}{9000} = 0.99$$

Thus, the mode of failure is a matrix failure.

By setting $E_2 = G_{12} = 0$ in the 90° -plies and reformulating [A], we have

$$[A] = \begin{bmatrix} 3.34 & 0.97 & 0 \\ 0.97 & 3.48 & 0 \\ 0 & 0 & 1.14 \end{bmatrix} \times 10^5 \; lb/in$$

Though not necessary, we continue to load the laminate with $N_x = 2063$ lb/in. At this load, the stresses in the laminas are obtained as

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\}_{0^\circ} = \left\{ \begin{array}{c} 134.55 \\ 0.11 \\ 0 \end{array} \right\} \times 10^3 psi$$

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\}_{\pm 45^\circ} = \left\{ \begin{array}{c} 49.87 \\ 4.72 \\ \mp 8.60 \end{array} \right\} \times 10^3 psi$$

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\}_{90^\circ} = \left\{ \begin{array}{c} -37.51 \\ 0 \\ 0 \end{array} \right\} \times 10^3 psi$$

The corresponding failure load for each lamina is calculated using the *Hill-Tsai criterion* as before. The results are

$$\begin{aligned} (N_{xcr})_{0^\circ} &= 4756 \; lb/in \\ (N_{xcr})_{\pm 45^\circ} &= 2605 \; lb/in \\ (N_{xcr})_{90^\circ} &= 17336 \; lb/in \end{aligned}$$

Thus, the second ply-failure-load is equal to 2605 lb/in. The mode of failure is easily identified as matrix failure (in-plane shear) in the ±45° -plies.

After modifying the $\overline{Q}_{ij}$ for the ±45° -plies, a new [A] matrix is obtained, and the laminar stress analysis continues. At $N_x = 2605 lb/in$, the stresses in the laminas are found as

$$\left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right\}_{0^\circ} = \left\{ \begin{array}{c} 195.16 \\ 0.61 \\ 0 \end{array} \right\} \times 10^3 psi$$

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{array}\right\}_{\pm 45^\circ} = \left\{\begin{array}{c}65.32\\ 0\\ 0\end{array}\right\} \times 10^3 psi$$

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{array}\right\}_{90^\circ} = \left\{\begin{array}{c}-64.71\\ 0\\ 0\end{array}\right\} \times 10^3 psi$$

Examination of these laminar stresses with the failure criterion indicates that the 0° -plies are to fail first at the load

$$N_{xcr} = 4129\ lb/in$$

and the mode of failure is that of fiber failure. Once there is fiber failure, the laminate is assumed to have failed. Thus, the laminate under consideration has the uniaxial tensile strength of $N_{xcr} = 4129 lb/in$.

Incremental Stiffness Reduction Model

To avoid the sudden jump in strain at ply failures, a model resembling the bilinear hardening rule in classical plasticity can be formulated. It is assumed that the reduced laminate stiffness governs only the incremental load-deformation relations, i.e.,

$$\left\{\begin{array}{c}\Delta N\\ -\\ \Delta M\end{array}\right\}_i = \left\{\begin{array}{c|c}\overline{A} & \overline{B}\\ - & -\\ \overline{B} & \overline{D}\end{array}\right\}_i \left\{\begin{array}{c}\Delta\varepsilon^\circ\\ -\\ \Delta\kappa\end{array}\right\}_i \tag{6.42}$$

where Δ indicates the increment, and $\overline{A}$, $\overline{B}$, and $\overline{D}$ are the reduced laminate stiffnesses after the i-th ply-failure. For $i = 0, A, B$, and D of the virgin laminate stiffnesses are recovered. Thus, between the i-th and $(i+1)$th ply-failures, the incremental laminate deformation can be obtained from (6.42) for a given load increment . The total load is obtained from adding the load increment to the critical load causing the ith ply-failure, that is,

$$\left\{\begin{array}{c}N\\ -\\ M\end{array}\right\}_i = \left\{\begin{array}{c}N_{cr}\\ -\\ Mcr\end{array}\right\}_i + \left\{\begin{array}{c}\Delta N\\ -\\ \Delta M\end{array}\right\}_i \tag{6.43}$$

Similarly, the total strains and curvatures of the laminate are given by

$$\left\{ \begin{array}{c} \epsilon^{\circ} \\ - \\ \kappa \end{array} \right\}_i = \left\{ \begin{array}{c} \epsilon^{\circ}_{cr} \\ - \\ \kappa_{cr} \end{array} \right\}_i + \left\{ \begin{array}{c} \Delta\varepsilon^{\circ} \\ - \\ \kappa \end{array} \right\}_i \tag{6.44}$$

This step-by-step analysis leads to a nonlinear laminate stress-strain curve shown in Fig. 6.7 . For monotonic loading, the stress-strain curve produced by this model seems reasonable. The dilemma is encountered when one attempts to establish a reasonable unloading path. If classical plasticity is followed, then unloading should follow the elastic unloading path AB (see Fig. 6.7) which is parallel to the initial slope of the stress-strain curve. In most fiber composites with brittle matrices, however, the unloading path more or less follows the path AO. Thus, the loading curve beyond the first ply-failure cannot be regarded as the hardened yield surface in the sense of continuum plasticity.

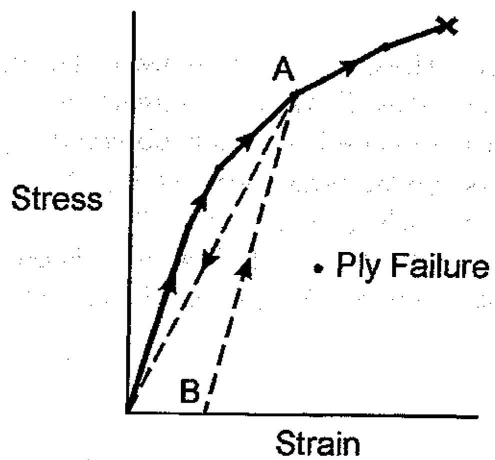


Fig. 6.7 Progressive failure of laminate

Sudden Failure Model

In fiber-dominated composite laminates, the laminate stiffness reduction due to progressive matrix failures is usually insignificant. This suggests that, in such laminates, the progressive stiffness reduction may be unnecessary and the laminate failure may be taken to coincide with the failure of the load carrying ply. Figure 6.8 shows the stress strain curves of a few fiber-dominated T300/5208 graphite/epoxy composite laminates under simple tension [6.5]. These results confirm our conjectures.

Consider Example 6.1 in which the 0○ -ply in the $[\pm 45/0/90]_S$, laminate is the controlling ply. From the laminate analysis results given by equation (b) the sudden failure model in conjunction with the *Hill-Tsai failure criterion* predicts the strength of the $[\pm 45/0/90]_S$ laminate to be

$$N_{xcr} = 4950 \ lb/in$$

Comparing this with N_{xcr}= 4129 lb/in predicted by the *parallel spring model*, it is evident that the sudden failure model tends to predict a higher laminate strength.

This model was evaluated experimentally by Kim [6.5] using T300/5208 graphite/epoxy composite laminates in conjunction with *Tsai-Wu failure criterion.* The unidirectional composite properties are

$$
\begin{aligned}
E_1 &= 20 \times 10^6\, psi\ (137.9\, GPa) \\
E_2 &= 1.4 \times 10^6\, psi\ (9.7\, GPa) \\
G_{12} &= 0.8 \times 10^6\, psi\ (5.5\, GPa) \\
\nu_{12} &= 0.3 \\
X &= -X' = 200\, ksi\ (1380\, MPa) \\
Y &= 6.96\, ksi\ (48\, MPa), Y' = -27.8\, ksi\ (-192\, MPa) \\
S &= 13.0\, ksi\ (90 MPa)
\end{aligned}
$$

Figure 6.9 presents the comparison between the theoretical and experimental strengths of $[0_2/\pm\theta]_S$ laminates. Fairly good agreement is noted. For these laminates, no delamination prior to laminate failure was observed.

Figure 6.10 shows similar results for $[0_2/\pm\theta/90]_S$. and $[0_2/90/\pm\theta]_S$ laminates. The theoretical results are fairly good except for $\theta \leq 45°$ in $[0_2/\pm\theta/90]_S$, laminates. In these laminates with $\theta \leq 45°$ extensive delamination is present which is an additional failure mode that is not accounted for by any of the three laminate failure models.

References

6.1 R Hill, *Mathematical Theory of Plasticity*, Oxford University Press, New York, 1950.

6.2 S. W. Tsai, "Strength Characteristics of Composite Materials," NASA CR-224, 1965.

6.3 S. W. Tsai and E. M. Wu, "A General Theory of Strength for Anisotropic Materials," *Journal of Composite Materials*, Vol. 5, 1971, pp. S8-80

6.4 Z. Hashin and A. Rotem, "A Fatigue Failure Criterion for Fiber Reinforced Materials," *Journal of Composite Materials*, Vol. 7, 1973, pp. 448-465.

6.5 R. Y. Kim, "In-Plane Tensile Strength of Multidirectional Composite Laminates," VDR-TR-81-84, University of Dayton Research Institute, Dayton, Ohio, August 1981.

Problems

6.1 The properties of the $AS4/3501-6$ graphite/epoxy composite are given as

$$\begin{aligned} E_1 &= 140GPa\ ,\ E_2 = 10GPa\ ,\ G_{12} = 7GPa \\ v_{12} &= 0.29 \\ X &= 2200MPa\ ,\ X' = -1700MPa \\ Y &= 60MPa\ ,\ Y' = -200MPa\ ,\ S = 100MPa \\ \alpha_1 &= -1 \times 10^{-6}/°C\ ,\ \alpha_2 = +26 \times 10^{-6}/°C \end{aligned}$$

a) Plot the failure envelopes for the composite under biaxial loading ($\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, $\sigma_{12} = 0$) using all five failure criteria

b) Plot the maximum uniaxial tensile stress σ_{xx} applied to off-axis composites versus the off-axis angle θ (from 0° to 90°) using maximum stress and Hill-Tsai criteria.

6. 2 Find the first ply failure and ultimate failure loads for the $AS4/3501-6\,[\pm 45°]_S$ laminate subjected to uniaxial tension N_x. Use the Separate Mode Failure Criterion.

6. 3 Assume that the stress-free temperature for a square $AS4/3501 - 6\,[0/90]_S$ composite laminate is 200°C. The edges of the laminate are fixed. Find the temperature at which first ply failure would occur according to the Maximum Stress Criterion.

Final

May 2nd

****** ORHTOTROPIC MATERIAL ****** \\

$$S = \begin{bmatrix} \frac{1}{Ex} & -\frac{vyx}{Ey} & -\frac{vzx}{Ez} & 0 & 0 & 0 \\ -\frac{vxy}{Ex} & \frac{1}{Ey} & -\frac{vzy}{Ez} & 0 & 0 & 0 \\ -\frac{vxz}{Ex} & -\frac{vyz}{Ey} & \frac{1}{Ez} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{Gyz} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{Gxz} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{Gxy} \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{(-1 + vyz\ vzy)\ Ex}{\%1}, & -\frac{(vyx + vyz\ vzx)\ Ex}{\%1}, & -\frac{(vyx\ vzy + vzx)\ Ex}{\%1}, & 0, & 0, & 0 \\ -\frac{(vxy + vxz\ vzy)\ Ey}{\%1}, & \frac{(-1 + vxz\ vzx)\ Ey}{\%1}, & -\frac{(vzy + vxy\ vzx)\ Ey}{\%1}, & 0, & 0, & 0 \\ -\frac{(vxy\ vyz + vxz)\ Ez}{\%1}, & -\frac{(vyz + vxz\ vyx)\ Ez}{\%1}, & \frac{(-1 + vxy\ vyx)\ Ez}{\%1}, & 0, & 0, & 0 \\ 0, & 0, & 0, & Gyz, & 0, & 0 \\ 0, & 0, & 0, & 0, & Gxz, & 0 \\ 0, & 0, & 0, & 0, & 0, & Gxy \end{bmatrix}$$

$\%1 := -1 + vyz\ vzy + vxy\ vyx + vxy\ vyz\ vzx + vxz\ vyx\ vzy + vxz\ vzx$

if Plane Stress (2D)

$Ez = vyz = vzy = vxz = vzx = 0$